

RANDOMNESS AND DIFFERENTIABILITY

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ABSTRACT. We characterize some major algorithmic randomness notions via differentiability of effective functions.

(1) We show that a real number $z \in [0, 1]$ is computably random if and only if every nondecreasing computable function $[0, 1] \rightarrow \mathbb{R}$ is differentiable at z .

(2) A real number $z \in [0, 1]$ is weakly 2-random if and only if every almost everywhere differentiable computable function $[0, 1] \rightarrow \mathbb{R}$ is differentiable at z .

(3) Recasting results of the constructivist Demuth (1975) in classical language, we show that a real z is Martin-Löf random if and only if every computable function of bounded variation is differentiable at z , and similarly for absolutely continuous functions.

We also use the analytic methods to show that computable randomness of a real is base invariant, and to derive preservation results for randomness notions.

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1. INTRODUCTION

The main thesis of this paper is that algorithmic randomness of a real is equivalent to differentiability of effective functions at the real. In more detail, for every major algorithmic randomness notion, one can provide a class of effective functions on the unit interval so that

(*) a real $z \in [0, 1]$ satisfies the randomness notion \Leftrightarrow
each function in the class is differentiable at z .

For instance, z is computably random \Leftrightarrow each computable nondecreasing function is differentiable at z , and z is Martin-Löf random \Leftrightarrow each computable function of bounded variation is differentiable at z . The second result was proved by Demuth [6], who used constructive language; we will reprove it here in the usual language using the first result relative to an oracle set.

1.1. A theorem of Lebesgue. Classically, to say that a property holds for a “random” real $z \in [0, 1]$ simply means that the reals failing the property form a null set. For instance, a well-known theorem of Lebesgue [13] states that every nondecreasing function $f: [0, 1] \rightarrow \mathbb{R}$ is differentiable at all reals z outside a null set (depending on f). That is, $f'(z)$ exists for a random real z in the sense specified above. Via Jordan’s result that each function of bounded variation is the difference of two nondecreasing functions (see, for instance, [1, Cor 5.2.3]), Lebesgue’s theorem can be extended to functions of bounded variation.

In most of the results of the type (*) above, the implication \Rightarrow can be seen as an effective form of Lebesgue's theorem. Before we make this precise, we will discuss algorithmic randomness, and computable functions on the unit interval.

1.2. Some preliminaries.

Algorithmic randomness. The idea in algorithmic randomness is to think of a real as random if it is in no *effective* null set. To specify an algorithmic randomness notion, one has to specify a type of effective null set, which is usually done by introducing a test concept. Failing the test is the same as being in the null set.

A hierarchy of algorithmic randomness notions has been developed, each one corresponding to certain aspects of our intuition. A central one is *Martin-Löf randomness*. A Σ_1^0 set $\mathcal{G} \subseteq [0, 1]$ has the form $\bigcup_m A_m$, where A_m is an open interval with dyadic rational endpoints obtained effectively from m . Let λ denote the usual Lebesgue measure on the unit interval. A *Martin-Löf test* is a sequence of uniformly Σ_1^0 sets $(\mathcal{G}_m)_{m \in \mathbb{N}}$ in the unit interval such that $\lambda \mathcal{G}_m \leq 2^{-m}$ for each m . The algorithmic null set it describes is $\bigcap_m \mathcal{G}_m$.

A notion implied by Martin-Löf randomness is computable randomness: a real is *computably random* if no computable betting strategy can win on its binary expansion (see Subsection 3.1 for detail).

A notion implying Martin-Löf randomness is weak 2-randomness. A Π_2^0 set (or effective G_δ set) is of the form $\bigcap_m \mathcal{G}_m$, where $(\mathcal{G}_m)_{m \in \mathbb{N}}$ is a sequence of uniformly Σ_1^0 sets. We call a real *weakly 2-random* if it is in no null Π_2^0 set. Compared to Martin-Löf randomness, the test notion is relaxed by replacing the condition $\forall m \lambda \mathcal{G}_m \leq 2^{-m}$ above by the weaker condition $\lim_m \lambda \mathcal{G}_m = 0$. For background on algorithmic randomness see Chapter 3 of [15] or [16].

Computable functions on the unit interval. Several definitions of computability for a function $f: [0, 1] \rightarrow \mathbb{R}$ have been proposed. In close analogy to the Church-Turing thesis, many (if not all) of them turned out to be equivalent. The common notion comes close to being a generally accepted formalization of computability for functions on the unit interval. Functions that are intuitively computable, such as e^x and \sqrt{x} , are computable in this formal sense. Computable functions in that sense are necessarily continuous; see the discussion in Weihrauch [21]. The generally accepted notion goes back to work of Grzegorzczuk and Lacombe from the 1950s, as discussed in Pour-El and Richards prior to [18, Def. A, p. 25]. In the same book [18, Def. C, p. 26] they give a simple condition equivalent to computability of f , which they call “effective Weierstrass”:

$f: [0, 1] \rightarrow \mathbb{R}$ is computable \Leftrightarrow there is an effective sequence $(P_n)_{n \in \mathbb{N}}$ of polynomials with rational coefficients such that $\|f - P_n\|_\infty \leq 2^{-n}$ for each n . This can be interpreted as saying that f is a computable point in a suitable computable metric space. See Subsection 2.2 for another characterization.

1.3. Results of type (*): the implication \Rightarrow .

(a) We will show in Theorem 4.1 that

a real $z \in [0, 1]$ is computably random \Rightarrow
each nondecreasing computable function is differentiable at z .

This is an effectivization of Lebesgue's theorem in terms of the concepts given above. Lebesgue's theorem is usually proved via Vitaly coverings. This method is non-constructive; a new approach is needed for the effective version.

- (b) The corresponding result of Demuth [6] involving Martin-Löf randomness and computable functions of bounded variation will be re-obtained as a corollary, using an effective form of Jordan's theorem.
- (c) For weak 2-randomness, we take the largest class of computable functions that makes sense in this setting: the a.e. differentiable computable functions. The implication \Rightarrow is obtained by observing that the points of nondifferentiability for any computable function is a Σ_3^0 set (i.e., an effective $G_{\delta\sigma}$ set). If the function is a.e. differentiable, this set is null, and hence cannot contain a weakly 2-random.

1.4. Results of type (*): the converse implication \Leftarrow .

The implication \Leftarrow in results of the form (*) is proved by contraposition. We simulate tests by non-differentiability of functions. Thus, given a test in the sense of the algorithmic randomness notion, one builds a computable function f on the unit interval such that, for each real z failing the test, $f'(z)$ fails to exist. We will provide direct, uniform constructions of this kind for weak 2-randomness (c) and Martin-Löf randomness (b). The computable function we build is a sum of "sawtooth functions". For computable randomness (a), the simulation is less direct, though still uniform. The results in more detail are as follows.

- (a) For each real z that is not computably random, there is a computable nondecreasing function f such that $\overline{D}f(z) = \infty$ (Theorem 4.1).
- (b) There is, in fact, a single computable function f of bounded variation such that $f'(z)$ fails to exist for all non-Martin-Löf random reals z (Lemma 5.5).
- (c) For each Π_2^0 null set there is an a.e. differentiable computable function f that is non-differentiable at any z in the null set (Theorem 5.1).

As mentioned above, (b) was already stated by Demuth [6, Example 2].

Classical counterparts of this implication exist. For instance, Zahorski [22] proved that each null G_δ set is the non-differentiability set of a monotonic Lipschitz function. For a recent proof, see Fowler and Preiss [10].

1.5. Classes of effective functions, and randomness notions.

The results mean that all major algorithmic randomness notions can now be matched with at least one class of effective functions on the unit interval in such a way that randomness is equivalent to differentiability. Surprisingly, the analytical properties of functions we use are the well-known ones from classical real analysis.

The matching is onto, but not 1-1: in a sense, randomness notions are coarser than classes of effective functions. Computable randomness is characterized not only by differentiability of nondecreasing computable functions, but also of computable Lipschitz functions [11]. Furthermore, as an effectiveness condition on functions, one can choose anything between computability in the sense discussed in Subsection 1.2 above, and the weaker

condition that $f(q)$ is a computable real (see Subsection 2.1), uniformly in a rational $q \in [0, 1]$. Several notions lying in between have received attention. One of them is Markov computability, which will be discussed briefly in Section 6.

A further well-studied algorithmic randomness notion is Schnorr randomness, still weaker than computable randomness (see, for instance, [15, Section 3.5]). A *Schnorr test* is a Martin-Löf test $(\mathcal{G}_m)_{m \in \mathbb{N}}$ such that $\lambda \mathcal{G}_m$ is a computable real uniformly in m . A real z is Schnorr random if $z \notin \bigcap_m \mathcal{G}_m$ for each Schnorr test $(\mathcal{G}_m)_{m \in \mathbb{N}}$.

To match this notion with a class of functions, one needs a notion of being effective that is stronger than the usual computability: the function f is a computable point in the Banach space $AC_0[0, 1]$ of absolutely continuous functions vanishing at 0 with the variation norm, where the computable structure (in the sense of [18, Ch. 2]) is given, for instance, by polynomials with rational coefficients. This means that for each n , one can determine a polynomial P_n with rational coefficients vanishing at 0 such that the variation of $f - P_n$ is at most 2^{-n} . By the effective version of a classical theorem from analysis (see, for instance, [3, Ch. 20]), this space is effectively isometric with the space $(\mathcal{L}_1[0, 1], \|\cdot\|_1)$, where the computable structure is also given by the polynomials with rational coefficients. The isomorphism is given by differentiation, and its inverse by the indefinite integral.

Recent results of J. Rute, and independently Pathak, Rojas and Simpson, can be restated as follows: z is Schnorr random iff each variation computable absolutely continuous function is differentiable at z . Freer, Kjos-Hanssen, and Nies [11] showed the analogous result for variation computable Lipschitz functions. See Nies' 2011 talk “Randomness and computable analysis: results and open questions” for more details.

The matching between algorithmic randomness notions and classes of effective functions is summarized in Figure 1.

1.6. Discussion. The results above indicate a rich two-way interaction between algorithmic randomness and analysis.

Analysis to randomness: Characterizations via differentiability can be used to better understand an algorithmic randomness notion. For instance, we will show that several randomness notions of reals are preserved under the maps $z \rightarrow z^\alpha$ where $\alpha \neq 0$ is a computable real. Furthermore, we show that computable randomness of a real is base invariant: it does not depend on the fact that we used the binary expansion of a real in its definition (Theorem 3.7 below).

Randomness to analysis: the results also yield a better understanding of the underlying classical theorems. They indicate that in the setting of Lebesgue's theorem (see Subsection 1.1), the exception sets for differentiability of nondecreasing functions are simpler than the exception sets for bounded variation functions. Furthermore, one can attempt to calibrate, in the sense of reverse mathematics, the strength of theorems saying that a certain function is a.e. well behaved. The benchmark principles have the form “for each oracle set X , there is a set R that is random in X ”, for some fixed algorithmic randomness notion. For Martin-Löf randomness, the principle

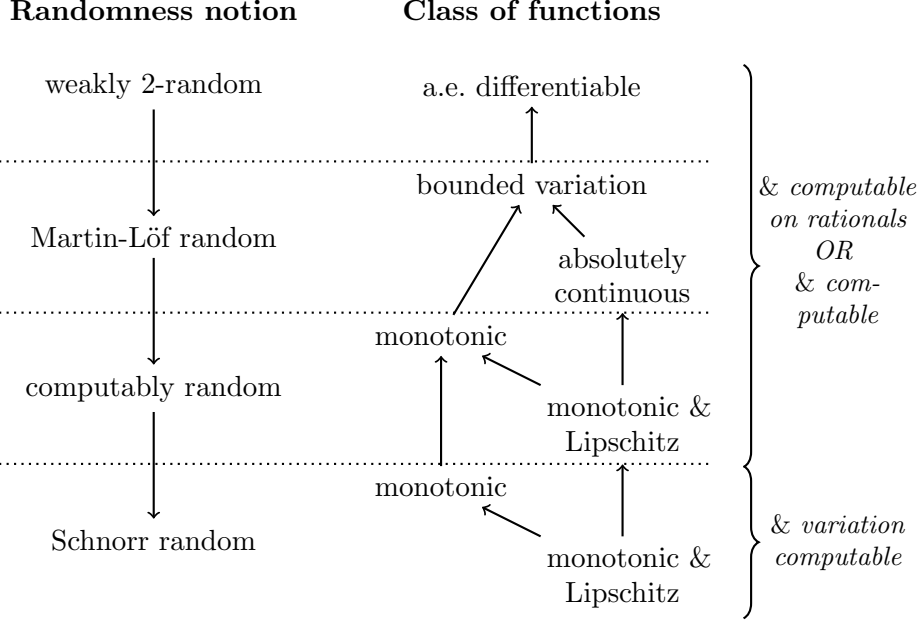


FIGURE 1. Randomness notions matched with classes of effective functions defined on $[0, 1]$ so that $(*)$ holds

above is called “weak weak König’s Lemma”. By recent work of Simpson and Pathak, this principle is sufficient to prove the Lebesgue differentiation theorem of Subsection 2.7 over a standard base theory called RCA_0 . A lot of work remains to be done here.

1.7. Structure of the paper. Section 2 provides background from computable analysis. Section 3 introduces computable randomness and shows its base invariance. The central Section 4 characterizes computable randomness in terms of differentiability of computable functions, and ends with some application on \mathcal{L}_1 -computable functions. Section 5 characterizes weak 2-randomness in terms of differentiability of computable functions, and provides the implication \Leftarrow in $(*)$ for Martin-Löf randomness. The final Section 6 extends the results to functions that are merely computable on the rationals, and to notions in between such as Markov computability which is introduced briefly. It also provides the implication \Rightarrow in $(*)$ for Martin-Löf randomness. The paper ends with some open questions and future directions.

2. BACKGROUND ON COMPUTABLE ANALYSIS

2.1. Computable reals. A sequence $(q_n)_{n \in \mathbb{N}}$ of rationals is called a *Cauchy name* if $|q_n - q_{n-1}| \leq 2^{-n}$ for each $n > 0$. If $\lim_n q_n = x$ we say that $(q_k)_{k \in \mathbb{N}}$ is a *Cauchy name* for x . Thus, q_n approximates x up to an error $|x - q_n|$ of at most 2^{-n} .

Each $x \in [0, 1]$ has a Cauchy name $(q_n)_{n \in \mathbb{N}}$ such that $q_0 = 0$, $q_1 = 1/2$, and each q_n is of the form $i2^{-n}$ for an integer i . Thus, if $n > 0$ then $q_n - q_{n-1} = a2^{-n}$ for some $a \in \Sigma = \{-1, 0, 1\}$. In this way a real x

corresponds to an element of Σ^ω . (This is a name for x in the *signed-digit representation* of reals; see [21].)

A real x is called *computable* if it has a computable Cauchy name. (Note that this definition is equivalent to the one previously given, that the binary expansion be computable as a sequence of bits; the binary expansion can, however, not be obtained uniformly from a Cauchy name, since one would need to know whether the real is a dyadic rational. The definition given here is more convenient to work with.)

A sequence $(x_n)_{n \in \mathbb{N}}$ of reals is *computable* if x_n is computable uniformly in n . That is, there is a computable double sequence $(q_{n,k})_{n,k \in \mathbb{N}}$ of rationals such that each x_n is a computable real as witnessed by its Cauchy name $(q_{n,k})_{k \in \mathbb{N}}$.

2.2. More on computable functions defined on the unit interval.

Let $f: [0, 1] \rightarrow \mathbb{R}$. The formal definition of computability closest to our intuition is perhaps the following: there is a Turing functional that with a Cauchy name $(q_n)_{n \in \mathbb{N}}$ of x as an oracle returns a Cauchy name of $f(x)$. (Thus, equivalent Cauchy names for an argument x yield equivalent Cauchy names for the value $f(x)$.) This condition means that given $k \in \mathbb{N}$, using enough of the sequence $(q_n)_{n \in \mathbb{N}}$ we can compute a rational p such that $|x - p| \leq 2^{-k}$.

It is often easier to work with the equivalent Definition A in Pour-El and Richards [18, p. 26].

Definition 2.1. *The function $f: [0, 1] \rightarrow \mathbb{R}$ is called computable if*

- (a) *for each computable sequence of reals $(x_k)_{k \in \mathbb{N}}$, the sequence $f(x_k)$ is computable, and*
- (b) *f is effectively uniformly continuous: there is a computable $h: \mathbb{N} \rightarrow \mathbb{N}$ such that $|x - y| < 2^{-h(n)}$ implies $|f(x) - f(y)| < 2^{-n}$ for each n .*

If f is effectively uniformly continuous, we can replace (a) by the following apparently weaker condition.

- (a') *for some computable sequence of reals $(v_i)_{i \in \mathbb{N}}$ that is dense in $[0, 1]$ the sequence $f(v_i)_{i \in \mathbb{N}}$ is computable.*

Typically, the sequence $(v_i)_{i \in \mathbb{N}}$ in (a') is an effective listing of the rationals in $[0, 1]$ without repetitions. To show (a') & (b) \Rightarrow (a), suppose $(x_n)_{n \in \mathbb{N}}$ is a computable sequence of reals as witnessed by the computable double sequence $(q_{n,k})_{n,k \in \mathbb{N}}$ of rationals (see Subsection 2.1). Let h be as in (b). We may assume that $h(p) \geq p$ for all p . To define the p -th approximation to $f(x_n)$, find i such that $|q_{n,h(p+1)} - v_i| < 2^{-h(p+1)}$. Output the p -th approximation r to $f(v_i)$. Observe that

$$|f(x_n) - r| \leq 2^{-p} + |f(x_n) - f(q_{n,h(p+1)})| + |f(q_{n,h(p+1)}) - f(v_i)| \leq 2^{-p+1},$$

as required.

An *index* for a computable function on the unit interval f is a pair consisting of a computable index for the double sequence $(q_{n,k})_{n,k \in \mathbb{N}}$ of rationals determining the values of f at the rationals, together with a computable index for h .

2.3. Computability for nondecreasing functions. We will frequently work with nondecreasing functions. Mere continuity and (a') are sufficient for such a function to be computable. This easy fact will be very useful later on.

Proposition 2.2. *Let g be a nondecreasing function. Suppose there is a computable dense sequence $(v_i)_{i \in \mathbb{N}}$ of reals in $[0, 1]$ such that the sequence of reals $g(v_i)_{i \in \mathbb{N}}$ is computable. Suppose that g is also continuous. Then g is computable.*

Proof. We analyze the usual proof that g is uniformly continuous in order to verify that g is effectively uniformly continuous, as defined in (b) of Definition 2.1. To define a function h as in (b), given n let $\epsilon = 2^{-n-2}$. Since g is nondecreasing, we can compute a covering of $[0, 1]$ by open sets of the form $(B_{\delta_i}(v_i))_{i \in F}$, where $F \subseteq \mathbb{N}$ is finite, and the δ_i are positive rationals, such that $d(x, v_i) < \delta_i \rightarrow d(g(x), g(v_i)) < \epsilon$. Let δ be the minimum distance of any pair of balls $B_{\delta_i}(v_i)$ with a disjoint closure. If $d(x, y) < \delta$ then choose $i, k \in F$ such that $x \in B_{\delta_i}(v_i)$ and $y \in B_{\delta_k}(v_k)$. These two balls are not disjoint, so we have $d(g(x), g(y)) < 4\epsilon = 2^{-n}$. Since we obtained δ effectively from n , we can determine $h(n)$ such that $2^{-h(n)} < \delta$. \square

Let $(v_n)_{n \in \mathbb{N}}$ list $I_{\mathbb{Q}}$ effectively without repetitions. For $\Sigma = \{-1, 0, 1\}$, via the representation $\Sigma^\omega \rightarrow [0, 1]$ of reals given above, we can name a function $h: I_{\mathbb{Q}} \rightarrow \mathbb{R}$ by some sequence $X \in \Sigma^\omega$. We let $X(\langle v_r, n \rangle)$ be the n -th entry in a name for $h(v_r)$.

2.4. Arithmetical complexity of sets of reals.

By an *open interval* in $[0, 1]$ we mean an interval of the form (a, b) , $[0, b)$, $(a, 1]$ or $[0, 1]$, where $0 \leq a \leq b \leq 1$.

A Σ_1^0 set in $[0, 1]$ is a set of the form $\bigcup_k A_k$ where $(A_k)_{k \in \mathbb{N}}$ is a computable sequence of open intervals with dyadic rational endpoints.

A Π_2^0 set has the form $\bigcap_m \mathcal{G}$ where the \mathcal{G}_m are Σ_1^0 sets uniformly in m .

The following well-known fact will be needed later.

Lemma 2.3. *Let $f: [0, 1] \rightarrow \mathbb{R}$ be computable. Then the sets $\{x: f(x) < p\}$ and $\{x: f(x) > p\}$ are Σ_1^0 sets, uniformly in a rational p .*

Proof. We verify the fact for sets of the form $\{x: f(x) < p\}$, the other case being symmetric. By (a') in Subsection 2.2, f is uniformly computable on the rationals in $[0, 1]$. Suppose $(q_k)_{k \in \mathbb{N}}$ is a computable Cauchy name for a real y . Then $y < s \Leftrightarrow \exists k [q_k < s - 2^{-k}]$. So we can uniformly in a rational s enumerate the set of rationals t such that $y = f(t) < s$.

Now let $h: \mathbb{N} \rightarrow \mathbb{N}$ be a function showing the effective uniform continuity of f in the sense of (b) of Subsection 2.2. To verify that $\mathcal{S} = \{x: f(x) < p\}$ is Σ_1^0 , we have to show that $\mathcal{S} = \bigcup_k A_k$ where $(A_k)_{k \in \mathbb{N}}$ is a computable sequence of open intervals with dyadic rational endpoints.

To define this sequence, for each n in parallel, do the following. Let $s_n = p - 2^{-n}$ and $\delta_n = 2^{-h(n)}$. When a dyadic rational t such that $f(t) < s_n$ is enumerated, add to the sequence $(A_k)_{k \in \mathbb{N}}$ the open interval $[0, 1] \cap (t - \delta_n, t + \delta_n)$.

Clearly $\bigcup_k A_k \subseteq \mathcal{S}$. For the converse inclusion, given $x \in \mathcal{S}$, choose n such that $f(x) + 2 \cdot 2^{-n} < p$, and choose a rational t such that x is in the open

interval $[0, 1] \cap (t - \delta_n, t + \delta_n)$. Then $f(t) < s_n$, so this interval is added to the sequence. \square

The proof is in fact uniform at a higher level: effectively in an index for f , one can obtain an index for the function mapping p to an index for the Σ_1^0 set $\{x: f(x) < p\}$.

2.5. Some notation and facts on differentiability. Unless otherwise mentioned, functions will have a domain contained in the unit interval.

For a function f , the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a, b) = \frac{f(a) - f(b)}{a - b}$$

Recall that if z is in the domain of f then

$$\begin{aligned} \overline{D}f(z) &= \limsup_{h \rightarrow 0} S_f(z, z + h) \\ \underline{D}f(z) &= \liminf_{h \rightarrow 0} S_f(z, z + h) \end{aligned}$$

Note that we allow the values $\pm\infty$. By the definition, a function f is differentiable at z if $\underline{D}f(z) = \overline{D}f(z)$ and this value is finite.

For $a < x < b$ we have

$$S_f(a, b) = \frac{x - a}{b - a} S_f(a, x) + \frac{b - x}{b - a} S_f(x, b). \quad (1)$$

This implies the following:

Fact 2.4. *Let $a < x < b$. Then*

$$\min\{S_f(a, x), S_f(x, b)\} \leq S_f(a, b) \leq \max\{S_f(a, x), S_f(x, b)\}.$$

Consider a set $V \subseteq \mathbb{R}$ that is dense in $[0, 1]$. If V is contained in the domain of a function f , we let

$$\begin{aligned} D^V f(x) &= \lim_{h \rightarrow 0+} \sup\{S_f(a, b): \\ &\quad a, b \in V \cap [0, 1] \text{ \& } a \leq x \leq b \text{ \& } 0 < b - a \leq h\} \\ D_V f(x) &= \lim_{h \rightarrow 0+} \inf\{S_f(a, b): \\ &\quad a, b \in V \cap [0, 1] \text{ \& } a \leq x \leq b \text{ \& } 0 < b - a \leq h\}. \end{aligned}$$

If $f(z)$ is defined, then Fact 2.4 implies that

$$\underline{D}f(z) \leq D_V f(z) \leq D^V f(z) \leq \overline{D}f(z). \quad (2)$$

The following lemma will be needed in Section 4. We say that z is in the *middle third* of an interval (a, b) if $a + (b - a)/3 \leq z \leq b - (b - a)/3$.

Lemma 2.5. *Suppose that $f: [0, 1] \rightarrow \mathbb{R}$ is continuous at z . If f is not differentiable at z , then this fact is witnessed on intervals that contain z in their middle third. More precisely, for $h > 0$ let*

$$\mathcal{I}_h = \{(a, b): 0 < b - a < h \text{ \& } z \text{ is in the middle third of } (a, b)\}.$$

Suppose that $v = \lim_{h \rightarrow 0} \{S_f(a, b): (a, b) \in \mathcal{I}_h\}$ exists. Then $f'(z) = v$.

Proof. The continuity of f at z implies that $f'(z)$ equals the limit of $S_f(a, b)$ over all open intervals (a, b) that contain z , as $b - a \rightarrow 0$. Take h and $t < s$ such that $t < S_f(a, b) < s$ for all $(a, b) \in \mathcal{I}_h$. Consider an interval (c, d) containing z such that $d - c < h/3$. We will prove that

$$5t - 4s < S_f(c, d) < 5s - 4t.$$

Note that we can take t and s to approach v as $h \rightarrow 0$, in which case both $5t - 4s$ and $5s - 4t$ also approach v . This implies that $f'(z) = v$.

Assume, without loss of generality, that z is closer to c than to d . Let $\delta = z - c$ and let $a_n = z - 2^{2n}\delta$ and $b_n = z + 2^{2n+1}\delta$ for all $n \in \omega$. So $a_0 = c$ and, for each n , the intervals (a_n, b_n) and (a_{n+1}, b_{n+1}) contain z in their middle third.

We may assume that $b_0 < d$, because otherwise z would be in the middle third of (c, d) . Take N such that $b_N < d \leq b_{N+1}$. Note that z is in the middle third of (a_{N+1}, d) because $b_{N+1} \geq d$. This interval is the longest we will consider. Note that $d - c > b_N - z = 2^{2N+1}\delta$, so

$$d - a_{N+1} = (d - z) + (z - a_{N+1}) < (d - c) + 2^{2N+2}\delta < 3(d - c) \leq h.$$

Therefore, all of the intervals that occur in the first four lines of the following estimates are in \mathcal{I}_h . We have

$$\begin{aligned} f(d) - f(c) &= (f(d) - f(a_{N+1})) - (f(b_N) - f(a_{N+1})) + (f(b_N) - f(a_N)) \\ &\quad - \cdots + \cdots - (f(b_0) - f(a_1)) + (f(b_0) - f(a_0)) \\ &< s(d - a_{N+1}) - t(b_N - a_{N+1}) + s(b_N - a_N) \\ &\quad - \cdots + \cdots - t(b_0 - a_1) + s(b_0 - a_0) \\ &= s(d - c) + s(a_0 - a_{N+1}) - t(2^{2N+3} - 2)\delta + s(2^{2N+2} - 1)\delta \\ &= s(d - c) + (s - t)(2^{2N+3} - 2)\delta \\ &< s(d - c) + 4(s - t)2^{2N+1}\delta \\ &< s(d - c) + 4(s - t)(d - c). \end{aligned} \tag{3}$$

This proves that $S_f(c, d) < 5s - 4t$. The lower bound $5t - 4s < S_f(c, d)$ is obtained in an analogous way. \square

2.6. Binary expansions. By a *binary expansion* of a real $x \in [0, 1]$ we will always mean the one with infinitely many 0s. Co-infinite sets of natural numbers are often identified with reals in $[0, 1]$ via the binary expansion. In this way, the product measure on Cantor space 2^ω is turned into the *uniform* (Lebesgue) measure on $[0, 1]$.

2.7. Other results from classical analysis and measure theory. Several theorems in classical real analysis say that a certain function is well-behaved almost everywhere. Being well-behaved can mean other things than being differentiable, although it is usually closely related. We give two examples. Their effective contents will be discussed in Subsection 4.4 and Subsection 6.3, respectively. As before, λ denotes the usual Lebesgue measure on the unit interval.

The first result is usually called the Lebesgue differentiation theorem.

Theorem 2.6. *Let $g: [0, 1] \rightarrow \mathbb{R}$ be integrable. Then λ -almost surely,*

$$g(z) = \lim_{r \rightarrow 0} \frac{1}{r} \int_z^{z+r} g d\lambda.$$

For a proof, see [1, Section 5.4].

The second result applies to arbitrary functions on the unit interval.

Theorem 2.7. *Let f be a function $[0, 1] \rightarrow \mathbb{R}$. Then λ -almost surely, the Denjoy alternative holds at z :*

$$\text{either } f'(z) \text{ exists, or } \overline{D}f(z) = \infty \text{ and } \underline{D}f(z) = -\infty.$$

For a proof of a more general Theorem due to Denjoy, Saks and Young, see [1, Thm. 5.8.12].

3. COMPUTABLE RANDOMNESS

3.1. Martingales. Schnorr [20] maintained that Martin-Löf randomness, being based on computably enumerable objects as tests, is already too powerful to be considered algorithmic. As a more restricted notion of a test, he proposed *computable betting strategies*, certain computable functions M from $2^{<\omega}$ to the non-negative reals. Let Z be an infinite sequence of bits. When the player has seen $\sigma = Z \upharpoonright_n$, she can make a bet q , where $0 \leq q \leq M(\sigma)$, on what the next bit $Z(n)$ is. If she is right, she gets q . Otherwise she loses q .

The formal concept corresponding to betting strategies is the following.

Definition 3.1. *A martingale is a function $2^{<\omega} \rightarrow \mathbb{R}_0^+$ such that the fairness condition*

$$M(\sigma 0) + M(\sigma 1) = 2M(\sigma) \tag{4}$$

holds for each string σ . M succeeds on a sequence of bits Z if $M(Z \upharpoonright_n)$ is unbounded.

Recall from Subsection 2.1 that a real number x is called computable if there is a computable Cauchy sequence $(q_n)_{n \in \mathbb{N}}$ of rationals such that $|x - q_n| \leq 2^{-n}$ for each n . A martingale $M: 2^{<\omega} \rightarrow \mathbb{R}_0^+$ is called *computable* if $M(\sigma)$ is a computable real uniformly in a string σ .

3.2. Introduction to computable randomness. The following notion goes back to [20].

Definition 3.2. *A sequence of bits Z is called computably random if no computable martingale succeeds on Z . A real $z \in [0, 1)$ is called computably random if its binary expansion is computably random.*

In fact, it suffices to require that no rational-valued martingale succeeds on the binary expansion ([20], also see [15, 7.3.8]). We mention some facts about computable randomness. For details, definitions, references and proofs see [15, Ch. 7].

Computable randomness lies strictly in between Martin-Löf and Schnorr randomness. Computably random sets can have a very slowly growing initial segment complexity, e.g., $K(Z \upharpoonright_n) \leq^+ 2 \log n$. Left-c.e. computably random

sets can be Turing incomplete. In fact they exist in each high c.e. degree. Lowness for computable randomness implies being computable.

There is a characterization of computable randomness by Downey and Griffiths [9] in terms of special Martin-Löf tests called “computably graded tests”, and a characterization by Day [5] via the growth of initial segment complexity measured in terms of so-called “quick process machines”.

3.3. The savings property.

Definition 3.3. *We say that a martingale M has the savings property if $M(\rho) \geq M(\sigma) - 2$ for any strings σ, ρ such that $\rho \succeq \sigma$.*

The following is well-known.

Proposition 3.4. *For each computable martingale L there is a computable martingale M with the savings property that succeeds on the same sequences as L .*

Proof. Arguing as in the solution to [15, Exercise 7.1.14]), we may assume that $L(\sigma) > 0$ for each $\sigma \in 2^{<\omega}$, and $L(\emptyset) < 1$. We may also assume that L is rational valued by a result of Schnorr (see [15, Prop. 7.3.8]). We let $M = G + E$, where $G(\sigma) \in \mathbb{N}$ is the balance of the “savings account”, and $E(\sigma)$ is the balance of the “checking account” at σ . The function E is a supermartingale (see [15, Section 7.2]) bounded by 2. It uses the same betting factors $L(\rho\hat{b})/L(\rho)$ as L for a string ρ and $b \in \{0, 1\}$, but in between subsequent bets it may transfer capital to the savings account.

For each string ρ , whenever $b \in \{0, 1\}$ and the betting results in a value $v > 1$ at $\rho\hat{b}$, we transfer 1 from the checking to the savings account, defining $G(\rho\hat{b}) = G(\rho) + 1$; in this case E has the capital $v - 1$ at the string $\rho\hat{b}$. If $v \leq 1$ we let $E(\rho\hat{b}) = v$ and $G(\rho\hat{b}) = G(\rho)$.

$M = G + E$ has the savings property because, if $\rho \succeq \sigma$ are strings, then

$$M(\rho) - M(\sigma) \geq E(\rho) - E(\sigma) \geq -2.$$

If L succeeds on Z then $\lim_n G(Z \upharpoonright_n) = \infty$, whence $\lim_n M(Z \upharpoonright_n) = \infty$. Since L is computable so is M , because L is rational valued and the arithmetical operations are effective on Cauchy names for reals. \square

In general, if M is a martingale, then we have the upper bound $M(\sigma) \leq 2^{|\sigma|}M(\emptyset)$ for each string σ . If M has the savings property, then in fact

$$M(\sigma) \leq 2|\sigma| + M(\emptyset). \quad (5)$$

For otherwise, there is $\tau\hat{i} \preceq \sigma$ for some $i \in \{0, 1\}$ such that $M(\tau\hat{i}) > M(\tau) + 2$, whence $M(\tau\hat{(1-i)}) < M(\tau) - 2$.

3.4. A correspondence between martingales and nondecreasing functions. Each martingale M determines a measure on the algebra of clopen sets by assigning $[\sigma]$ the value $2^{-|\sigma|}M(\sigma)$. Via Carathéodory’s extension theorem this measure can be extended to a Borel measure in Cantor Space. We say that M is *atomless* if this measure is atomless, i.e., has no point masses. (This means that there is no $Z \in 2^\omega$ such that M eventually always

doubles its capital along Z .) If the measure is atomless, via the binary expansion of reals (Subsection 2.6) we can view it also as a Borel measure μ_M on $[0, 1]$. Thus, μ_M is determined by the condition

$$\mu_M([0.\sigma, 0.\sigma + 2^{-|\sigma|})) = 2^{-|\sigma|}M(\sigma).$$

We use the equality above to establish a relationship between atomless martingales and nondecreasing continuous functions, as follows.

Atomless martingales to nondecreasing continuous functions on $[0, 1]$. Given an atomless martingale M , let $\text{Fcn}(M)$ be the distribution function of the associated measure. That is,

$$\text{Fcn}(M)(x) = \mu_M[0, x].$$

Then $\text{Fcn}(M)$ is nondecreasing and continuous since the measure is atomless. Hence it is determined by its values on the rationals.

In the following let $I_{\mathbb{Q}} = [0, 1] \cap \mathbb{Q}$.

Nondecreasing functions with domain containing $I_{\mathbb{Q}}$ to martingales. If f is a nondecreasing function with a domain containing $I_{\mathbb{Q}}$, let

$$\text{Mart}(f)(\sigma) = S_f(0.\sigma, 0.\sigma + 2^{-|\sigma|}) = (f(0.\sigma + 2^{-|\sigma|}) - f(0.\sigma))/2^{-|\sigma|}.$$

For instance, $M(10) = S_f(\frac{1}{2}, \frac{3}{4})$, and $M(11) = S_f(\frac{3}{4}, 1)$. To see that $M = \text{Mart}(f)$ is a martingale, fix σ . Let $a = 0.\sigma, x = 0.\sigma + 2^{-|\sigma|-1}$, and $b = 0.\sigma + 2^{-|\sigma|}$. By the averaging condition on slopes in (1) we have

$$M(\sigma) = S_f(a, b) = S_f(a, x)/2 + S_f(x, b)/2 = M(\sigma 0)/2 + M(\sigma 1)/2.$$

Fact 3.5. *The transformations defined above induce a correspondence between atomless martingales and nondecreasing continuous functions on $[0, 1]$ that vanish at 0. In particular:*

- (i) *Let M be an atomless martingale. Then $\text{Mart}(\text{Fcn}(M)) = M$.*
- (ii) *Let f be a nondecreasing continuous function on $[0, 1]$ such that $f(0) = 0$. Then $\text{Fcn}(\text{Mart}(f)) = f$.*

Proof. (i) Let $f = \text{Fcn}(M)$. For each string σ we have

$$\text{Mart}(f)(\sigma) = S_f(0.\sigma, 0.\sigma + 2^{-|\sigma|}) = 2^{|\sigma|}\mu_M[0.\sigma, 0.\sigma + 2^{-|\sigma|}) = M(\sigma).$$

(ii) Let $M = \text{Mart}(f)$. Let μ be the measure on $[0, 1]$ such that $\mu[0, x) = f(x)$ for each x . Then $M(\sigma) = 2^{|\sigma|}\mu[0.\sigma, 0.\sigma + 2^{-|\sigma|})$ for each σ . Hence $\mu_M = \mu$ and $\text{Fcn}(M)(x) = \mu_M[0, x) = f(x)$ for each x . \square

Recall the definition of $D_V(z)$ from Subsection 2.5.

Theorem 3.6. *Suppose M is a martingale with the savings property (see Subsection 3.1). Let $g = \text{Fcn}(M)$. Suppose $z \in [0, 1]$ is not a dyadic rational. Then the following are equivalent:*

- (i) *M succeeds on the binary expansion of z .*
- (ii) *$\underline{D}g(z) = \infty$.*
- (iii) *$D_{\mathbb{Q}}g(z) = \infty$.*

In fact, the proof will show that the implications (ii)→(iii)→(i) do not rely on the hypothesis that M has the savings property. Of course we still need that M is atomless, to ensure that $\text{Fcn}(M)$ is defined.

Proof. Note that, since $z \in [0, 1)$ is not a dyadic rational, its binary expansion Z is unique.

(ii) → (iii). This is immediate because, by (2) in Subsection 2.5, we have $\underline{D}g(z) \leq D_{\mathbb{Q}}g(z)$.

(iii)→(i). Given $c > 0$, choose n such that $S_g(p, q) \geq c$ whenever p, q are rationals, $p \leq z \leq q$, and $q - p \leq 2^{-n}$. Let $\sigma = Z \upharpoonright_n$. Then we have $z \in [0.\sigma, 0.\sigma + 2^{-n}]$, and the length of this interval is 2^{-n} . Hence $M(\sigma) \geq c$.

(i) → (ii). We show that for each $r \in \mathbb{N}$ there is $\epsilon > 0$ such that $0 < |h| < \epsilon$ implies $g(z + h) - g(z) > rh$. This implies that $\underline{D}g(z) = \infty$.

Note that the binary expansion Z of z has infinitely many 0s and infinitely many 1s. Since M has the savings property, there is $i \in \mathbb{N}$ such that $Z(i) = 0$, $Z(i + 1) = 1$, and for $\sigma = Z \upharpoonright_i$, we have $\forall \tau M(\sigma\tau) \geq r$. Let $j > i$ be least such that $Z(j) = 0$. Let $\epsilon = 2^{-j-1}$. If $0 < |h| < \epsilon$ then the binary expansion of $z + h$ extends σ . If $h > 0$, this is because $z + 2^{-j-1} < 0.\sigma 1$. If $h < 0$, then adding h to z can at worst change the bit $Z(i + 1)$ from 1 to 0.

Let $W \subseteq 2^{<\omega}$ be a prefix free set of strings such that $[W]^\prec$ is identified with the open interval $(z, z + h)$ in case $h > 0$, and $[W]^\prec$ is identified with $(z + h, z)$ in case $h < 0$. All the strings in W extend σ . So we have in case $h > 0$,

$$g(z + h) - g(z) = \mu_M(z, z + h) = \sum_{\sigma \in W} M(\sigma)2^{-|\sigma|} \geq r \sum_{\sigma \in W} 2^{-|\sigma|} = rh,$$

and in case $h < 0$

$$g(z) - g(z + h) = \mu_M(z + h, z) = \sum_{\sigma \in W} M(\sigma)2^{-|\sigma|} \geq r \sum_{\sigma \in W} 2^{-|\sigma|} = -rh.$$

In either case we have $(g(z + h) - g(z))/h \geq r$. \square

3.5. Computable randomness is base-invariant. We give a first application of the analytical view of algorithmic randomness. This subsection will not be needed later on.

If the definition of a randomness notion for Cantor space is based on measure, it can be transferred right away to the reals in $[0, 1]$ by the correspondence in Subsection 2.6.

Among the notions in the hierarchy mentioned in the introduction, computable randomness is the only one not directly defined in terms of measure. We argue that computable randomness of a real is independent of the choice of expansions in base 2. We will use that the condition (iii) in Theorem 3.6 is base-independent. First, we give the relevant definitions.

Let $k \geq 2$. We can generalize the definition of martingales to the case that the domain consists of the strings of numbers in $\{0, \dots, k - 1\}$. Thus, a *martingale for base k* is a function

$$M: \{0, \dots, k - 1\}^{<\omega} \rightarrow \mathbb{R}_0^+$$

with the fairness condition $\sum_{i=0}^{k-1} (M(\sigma i) - M(\sigma)) = 0$, or equivalently,

$$\sum_{i=0}^{k-1} M(\sigma i) = kM(\sigma).$$

(An example is repeatedly playing a simple type of lottery, where k is the number of possible draws. We have seen a string σ of draws. We bet an amount $q \leq M(\sigma)$ on a certain draw; if we are right we get $(k-1) \cdot q$, otherwise we lose q .)

The theory of Subsections 3.3 and 3.4 can be developed more generally for martingales M in base k . Such an M induces a measure μ_M via $\mu_M([\sigma]) = M(\sigma)k^{-|\sigma|}$. As before, we call M atomless if μ_M is atomless as a measure. The remarks on the savings property after Definition 3.3, including (5), remain true. We have a transformation Fcn turning an atomless martingale in base k into a nondecreasing continuous function on $[0, 1]$ at 0, the distribution function of μ_M . There is an inverse transformation Mart^k turning such a function f into a martingale in base k via

$$\text{Mart}^k(f)(\sigma) = S_f(0.\sigma, 0.\sigma + k^{-|\sigma|}).$$

We call a sequence Z of numbers in $\{0, \dots, k-1\}$ *computably random in base k* if no computable martingale in base k succeeds on Z . Let us temporarily say that a real $z \in [0, 1)$ is *computably random in base k* if its base k expansion (with infinitely many entries $\neq k-1$) is computably random in base k .

Theorem 3.7. *Let $z \in [0, 1)$. Let $k, r \geq 2$ be natural numbers. Then z is computably random in base $k \Rightarrow z$ is computably random in base r .*

Proof. We may assume z is irrational. Let Z be the base k expansion, and let Y be the base r expansion of z . Suppose Y is not computably random in base r . Then some computable martingale M in base r with the savings property succeeds on Y . By (5) for base r we have $M(\sigma) \leq 2|\sigma| + O(1)$, whence M is atomless. Hence μ_M is defined and the associated distribution function,

$$f = \text{Fcn}(M)$$

is continuous. Clearly, $f(q)$ is uniformly computable for any rational $q \in [0, 1]$ of the form ir^{-n} , $i \in \mathbb{N}$. Hence, by Proposition 2.2, f is computable. Therefore the martingale in base k corresponding to f , namely

$$N = \text{Mart}^k(f),$$

is atomless and computable.

The proof of (i) \rightarrow (ii) in Theorem 3.6 works for base r : replace 2 by r , and replace the digits 0,1 by digits $b < c < r$ that both occur infinitely often in the r -ary expansion of z (these exist because z is irrational). So, since M has the savings property, we have $\underline{D}f(z) = \infty$. Note that $f = \text{Fcn}(N)$ by Fact 3.5 in base k . Hence by (ii) \rightarrow (i) of the same Theorem 3.6, but for base k , the computable martingale N succeeds on Z . \square

4. COMPUTABLE RANDOMNESS AND DIFFERENTIABILITY

We characterize computable randomness in terms of differentiability.

Theorem 4.1. *Let $z \in [0, 1]$. Then the following are equivalent:*

- (i) *z is computably random.*
- (ii) *Each computable nondecreasing function $f: [0, 1] \rightarrow \mathbb{R}$ is differentiable at z .*
- (iii) *Each computable nondecreasing function $g: [0, 1] \rightarrow \mathbb{R}$ satisfies $\overline{D}g(z) < \infty$.*
- (iv) *Each computable nondecreasing function $g: [0, 1] \rightarrow \mathbb{R}$ satisfies $\underline{D}g(z) < \infty$.*

Proof. The implications (ii) \rightarrow (iii) \rightarrow (iv) are trivial. For the implication (iv) \rightarrow (i), suppose that z is not computably random.

If z is rational, we can let $g(x) = 1 - \sqrt{z - x}$ for $x \leq z$ and $g(x) = 1$ for $x > z$. Clearly g is nondecreasing and $\underline{D}g(z) = \infty$. Since z is rational, g is uniformly computable on the rationals in $[0, 1]$. Hence g is computable by Proposition 2.2.

Now suppose otherwise z is irrational. Let the bit sequence Z correspond to the binary expansion of z . By Prop. 3.4, there is a computable martingale M with the savings property such that $\lim_n M(Z \upharpoonright_n) = \infty$. Let $g = \text{Fcn}(M)$. Then $\underline{D}g(z) = \infty$ by Theorem 3.6.

By the savings property of M and its consequence (5), we have $M(\sigma) = o(2^{|\sigma|})$, whence the measure μ_M is atomless. Hence the associated distribution function $g = \text{Fcn}(M)$ is continuous. Clearly $g(q)$ is uniformly computable on the dyadic rationals in $[0, 1]$. Then, once again by Proposition 2.2, we may conclude that g is computable.

It remains to prove the implication (i) \rightarrow (ii).

We actually prove (i) \rightarrow (iii) \rightarrow (ii). We begin with (iii) \rightarrow (ii) which relies on the algebraic Lemma 4.2 below. Thereafter, we will obtain (i) \rightarrow (iii) by a further application of that lemma.

4.1. Proof of (iii) \rightarrow (ii).

4.1.1. *Bettings on rational intervals.* For the rest of this proof, all intervals will be closed with distinct rational endpoints. If $A = [a, b]$ we write $|A|$ for the length $b - a$. To say that intervals are *disjoint* means they are disjoint on $\mathbb{R} \setminus \mathbb{Q}$. A *basic dyadic interval* has the form $[i2^{-n}, (i+1)2^{-n}]$ for some $i \in \mathbb{Z}, n \in \mathbb{N}$.

We prove the contraposition $\neg(\text{ii}) \rightarrow \neg(\text{iii})$. Suppose a computable nondecreasing function f is not differentiable at z . We will eventually define a computable nondecreasing function g such that $\overline{D}g(z) = \infty$. We may assume f is increasing after replacing f by the function $x \mapsto f(x) + x$. If $\underline{D}f(z) = \infty$ we are done by letting $g = f$. Otherwise, we have

$$0 \leq \underline{D}f(z) < \overline{D}f(z).$$

The nondecreasing computable function g is defined in conjunction with a betting strategy Γ . Instead of betting on strings, the strategy bets on nodes in a tree of rational intervals A . The root is $[0, 1]$, and the tree is ordered

by reverse inclusion. This strategy Γ proceeds from an interval A to sub-intervals A_k which are its successors on the tree. It maps these intervals to non-negative reals representing the capital at that interval. If the tree consists of the basic dyadic sub-intervals of $[0, 1]$, we have essentially the same type of betting strategy as before. However, it will be necessary to consider a more complicated tree where nodes have infinitely many successors.

We define the nondecreasing function g in such a way that at a node $A = [a, b]$, the capital is the slope:

$$\Gamma(A) = S_g(a, b) = \frac{g(b) - g(a)}{b - a}. \quad (6)$$

Thus, initially we define g only on the endpoints of intervals in the tree, which will form a dense sequence of rationals in $[0, 1]$ with an effective listing. Thereafter we will use Proposition 2.2 to extend g to all reals in the unit interval.

4.1.2. *The Doob strategy.* One idea in our proof is taken from the proof of the fact that $\lim_n M(Z \upharpoonright_n)$ exists for each computably random sequence Z and each computable martingale M : otherwise, there are rationals β, γ such that

$$\liminf_n M(Z \upharpoonright_n) < \beta < \gamma < \limsup_n M(Z \upharpoonright_n).$$

In this case one defines a new computable betting strategy G on strings that succeeds on Z . On each string, G is either in the betting state, or in the non-betting state. Initially it is in the betting state. In the betting state G bets with the same factors as M (i.e., $G(\sigma a)/G(\sigma) = M(\sigma a)/M(\sigma)$ for the current string σ and each $a \in \{0, 1\}$), until M 's capital exceeds γ . From then on, G does not bet until M 's capital is below β . On the initial segments of Z , the strategy G goes through infinitely many state changes; each time it returns to the non-betting state, it has multiplied its capital by γ/β . Note that this is an effective version of the technique used to prove the first Doob martingale convergence theorem.

If $A = [x, y]$, for the slope of f we use the shorthand

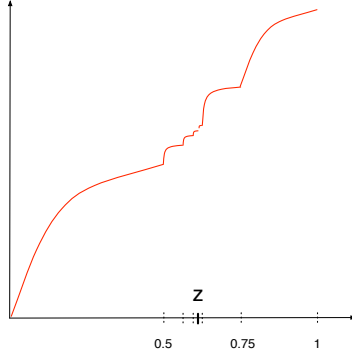
$$S_f(A) = S_f(x, y).$$

Given $z \in [0, 1] - \mathbb{Q}$, let A_n be the basic dyadic interval of length 2^{-n} containing z . A simple scenario is that our case assumption $\underline{D}f(z) < \overline{D}f(z)$ becomes apparent on these basic dyadic intervals:

$$\liminf_n S_f(A_n) < \beta < \gamma < \limsup_n S_f(A_n)$$

for some rationals $\beta < \gamma$. In this case, we may carry out the Doob strategy for the martingale M given by $M(\sigma) = S_f([0, \sigma, 0, \sigma + 2^{-|\sigma|}])$, in other words $M = \text{Mart}(f)$, and view it as a betting strategy on nodes in the tree of basic dyadic intervals.

Unfortunately, this scenario is too simple. For instance, if z is a computable irrational, the function indicated below satisfies $\underline{D}f(z) < \overline{D}f(z)$, but $S_f(A_n) = 1$ for each n .



We now describe how to deal with the general situation. For $p, q \in \mathbb{Q}$, $p > 0$, we say that an interval is a (p, q) -interval if it is the image of a basic dyadic interval under the affine transformation $y \rightarrow py + q$. Thus, a (p, q) -interval has the form

$$[pi2^{-n} + q, p(i+1)2^{-n} + q]$$

for some $i \in \mathbb{Z}, n \in \mathbb{N}$.

We will show in Lemma 4.3 that there are rationals p, q and r, s such that

$$\liminf_{\substack{|A| \rightarrow 0 \\ A \text{ is } (p,q)\text{-interval} \\ z \in A}} S_f(A) < \limsup_{\substack{|B| \rightarrow 0 \\ B \text{ is } (r,s)\text{-interval} \\ z \in B}} S_f(B). \quad (7)$$

The strategy Γ is in the betting state on (p, q) intervals in the tree of intervals, and in the non-betting state on (r, s) -intervals. For each state, it proceeds exactly like the Doob strategy in the corresponding state. In addition, when Γ switches state, the current interval is split into intervals of the other type (usually, into infinitely many intervals). Nonetheless, the other state takes effect immediately. So, in the betting state, we have to immediately bet on all the components of this (usually) infinite splitting.

4.1.3. The algebraic part. We will show in Lemma 4.3 that rationals as in (7) above exist. For this we need a further lemma. Suppose L is a finite set of rationals. An interval is called an L -interval if it is a (p, q) -interval for some $p, q \in L$.

Lemma 4.2. *For each rational $\alpha > 1$, we can effectively determine a finite set L of rationals such that for each interval $[x, y]$, $0 < x < y < 1$, there are L -intervals A, B as follows:*

$$\begin{aligned} [x, y] \subset A \quad &\& \quad \frac{|A|}{y-x} < \alpha, \\ B \subset [x, y] \quad &\& \quad \frac{y-x}{|B|} < \alpha. \end{aligned}$$

Informally, we can approximate the given interval from above and from below by L -intervals within a “precision factor” of α . We defer the proof of the lemma to Subsection 4.2.

By the hypothesis that $f'(z)$ does not exist and Lemma 2.5, we can choose rationals $\tilde{\beta} < \tilde{\gamma}$ such that

$$\begin{aligned}\tilde{\gamma} &< \limsup_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \text{ \& } z \in \text{middle third of } (x, y)\}, \\ \tilde{\beta} &> \liminf_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \text{ \& } z \in \text{middle third of } (x, y)\}\end{aligned}$$

Let α be a rational such that $1 < \alpha < 4/3$ and $\alpha^3 \tilde{\beta} < \tilde{\gamma}$. Let β, γ be rationals such that

$$\tilde{\beta}\alpha < \beta < \beta\alpha < \gamma < \gamma\alpha < \tilde{\gamma}.$$

Lemma 4.3. *There are rationals p, q, r, s , such that $p, r > 0$ and*

$$\begin{aligned}\gamma &< \limsup_{h \rightarrow 0} \{S_f(A) : A \text{ is } (p, q)\text{-interval \& } |A| \leq h \text{ \& } z \in A\}, \\ \beta &> \liminf_{h \rightarrow 0} \{S_f(B) : B \text{ is } (r, s)\text{-interval \& } |B| \leq h \text{ \& } z \in B\}.\end{aligned}$$

Proof. Let L be as in Lemma 4.2. For the first inequality, we use the first line in Lemma 4.2. We actually do not need the condition on middle thirds in the definition of $\tilde{\gamma}$.

Let $h > 0$ be given. Choose reals $x < y$, where $x \leq z \leq y$, such that $y - x < h/\alpha$ and $S_f(x, y) > \tilde{\gamma}$. By Lemma 4.2 there is an L -interval $A = [u, v]$ such that $[x, y] \subseteq A$ and $|A|/(y - x) < \alpha$. Then, since f is nondecreasing and $u - v < \alpha(y - x)$, we have

$$S_f(A) = \frac{f(v) - f(u)}{v - u} \geq \frac{f(y) - f(x)}{v - u} > \frac{f(y) - f(x)}{(y - x)\alpha} > \tilde{\gamma}/\alpha > \gamma.$$

Since L is finite, we can now pick a single pair of rationals $p, q \in L$ which works for arbitrary small $h > 0$, as required.

For the second inequality, the argument is similar, based on the second line in Lemma 4.2. However, we now need the condition on middle thirds in the definition of $\tilde{\beta}$, because when we replace an interval $[x, y]$ by a subinterval B that is an (r, s) -interval, we also want to ensure that $z \in B$.

Given $h > 0$, choose reals $x < y$, where $x \leq z \leq y$ and z is in the middle third of $[x, y]$, such that $y - x < h/\alpha$ and $S_f(x, y) < \tilde{\beta}$. By Lemma 4.2 there is an L -interval $B = [u, v]$ such that $B \subseteq [x, y]$ and $(y - x)/|B| < \alpha$. Since $\alpha < 4/3$ and z is in the middle third of $[x, y]$, we have $z \in B$. Similar to the estimates above, we have

$$S_f(B) \leq \frac{f(y) - f(x)}{v - u} \leq \frac{f(y) - f(x)}{(y - x)/\alpha} = \alpha S_f(x, y) \leq \alpha \tilde{\beta} < \beta.$$

□

4.1.4. *Definition of g on a dense set, and the strategy Γ .* In the following fix p, q, r, s as in Lemma 4.3. We will define an infinitely branching tree of intervals. As explained above, on each node A in this tree, the strategy is either

- in a *betting state*, betting on smaller and smaller (p, q) -sub-intervals of A , or
- in a *non-betting state*, processing smaller and smaller (r, s) -sub-intervals of A , but without betting.

The root of the tree is $A = [0, 1]$. Initially let $g(0) = 0$ and $g(1) = 1$ (hence $\Gamma(A) = 1$), and put the strategy Γ into the betting state. Let $K \in \mathbb{N}$ be large enough that the inequalities in Lemma 4.3 still hold with $\gamma + 2^{-K}$ instead of γ , and with $\beta - 2^{-K}$ instead of β , respectively. We also require that K is large enough that $(\gamma - 2^{-K})/(\beta + 2^{-K}) \geq \alpha$. For a real x named by a Cauchy sequence as in Subsection 2.1, we let x_K denote the K -th term of that sequence. Thus $|x - x_K| \leq 2^{-K}$.

Suppose $A = [a, b]$ is an interval such that $\Gamma(A)$ has already been defined. By hypothesis $S_f(A)$ is a computable real uniformly in A . Proceed according to the case that applies.

(I): Γ is in the betting state on A .

(I.a) $S_f(A)_K \leq \gamma$. If Γ has just entered the betting state on A , let

$$A = \bigsqcup_k A_k$$

where the A_k form an effective sequence of (p, q) -intervals that are disjoint (on $[0, 1] \setminus \mathbb{Q}$). Otherwise, split $A = A_0 \cup A_1$ into disjoint intervals of equal length.

The function g interpolates between a and b with a growth proportional to the growth of f : if $v \in (a, b)$ is an endpoint of a new interval, define

$$g(v) = g(a) + (g(b) - g(a)) \frac{f(v) - f(a)}{f(b) - f(a)}.$$

Continue the strategy on each sub-interval.

(I.b) $S_f(A)_K > \gamma$. Switch to the non-betting state on A and goto (II).

(II) Γ is in the non-betting state on A .

(II.a) $S_f(A)_K \geq \beta$. If Γ has just entered the non-betting state on A , let

$$A = \bigsqcup_k A_k$$

where the A_k form an effective sequence of (r, s) -intervals that are disjoint on $[0, 1] \setminus \mathbb{Q}$, and further, $2|A_k| \leq |A|$ for each k . Otherwise split $A = A_0 \cup A_1$ into disjoint intervals of equal length.

If $v \in (a, b)$ is an endpoint of a new interval, then g interpolates linearly: let

$$g(v) = g(a) + (g(b) - g(a)) \frac{v - a}{b - a}.$$

Continue the strategy on each sub-interval.

(II.b) $S_f(A)_K < \beta$. Switch to the betting state on A and goto (I).

4.1.5. *The verification.* If the strategy Γ , processing an interval $A = [a, b]$ in the betting state, chooses a sub-interval $[c, d]$, then

$$g(d) - g(c) = (g(b) - g(a)) \frac{f(d) - f(c)}{f(b) - f(a)}.$$

Dividing this equation by $d - c$ and recalling the definition of the Γ -values in (6), we obtain

$$\Gamma([c, d]) = \Gamma([a, b]) \frac{S_f(c, d)}{S_f(a, b)}. \quad (8)$$

The purpose of the following two claims is to extend g to a computable function on $[0, 1]$. For the rest of the proof, we will use the shorthand

$$g[A] = g(b) - g(a)$$

for an interval on the tree $A = [a, b]$. Recall that we write $S_g(A)$ for the slope $S_g(a, b)$. Thus $\Gamma(A) = S_g(A) = g[A]/|A|$.

Claim 4.4. *Let $x \in [0, 1]$. Let $B_0 \supset B_1 \supset \dots$ be an infinite path on the tree of intervals. Then $\lim_m g[B_m] = 0$.*

We consider the states of the betting strategy Γ as it processes the intervals $A = B_m$.

Case 1: Γ changes its state only finitely often when processing the intervals B_m .

If Γ is eventually in a non-betting state then clearly $\lim_m g[B_m] = 0$. Suppose otherwise, that is, Γ is eventually in a betting state. Suppose further that Γ enters the betting state for the last time when it defines the interval $A = B_{m^*}$. Then for all $m \geq m^*$, by (8) and the case hypothesis $S_f(B_m)_K \leq \gamma$, we have

$$\Gamma(B_m) = \Gamma(B_{m^*}) \frac{S_f(B_m)}{S_f(B_{m^*})} \leq (\gamma + 2^{-K}) \frac{\Gamma(B_{m^*})}{S_f(B_{m^*})} =: C.$$

Hence $g[B_m] = \Gamma(B_m) \cdot |B_m| \leq C|B_m|$.

Case 2: Γ changes its state infinitely often when processing the intervals B_m .

Let B_{m_i} be the interval A processed when the strategy is for the i -th time in a betting state at (I.b). Note that $g[B_{m_{i+1}}] \leq g[B_{m_i}]/2$ because at (II.a) we chose all the splitting components A_k at most half as long as the given interval A . Of course, by monotonicity of g we have $g[B_{m+1}] \leq g[B_m]$ for each m . Thus, $g[B_m] \leq 2^{-i}$ for each $m > m_i$. This completes the proof of the claim.

Claim 4.5. *The function g can be extended to a computable function on $[0, 1]$.*

Let V be the set of endpoints of intervals on the tree. Clearly V is dense in $[0, 1]$. For $x \in [0, 1]$ let

$$\begin{aligned} \underline{g}(x) &= \sup\{g(v) : v < x, v \in V\} \\ \bar{g}(x) &= \inf\{g(w) : w > x, w \in V\}. \end{aligned}$$

We show that $\underline{g}(x) \geq \bar{g}(x)$. Since g is nondecreasing on V , this will imply that $\underline{g} = \bar{g}$ is a continuous extension of g .

There is an infinite path $B_0 \supset B_1 \supset \dots$ on the tree of intervals such that $x \in \bigcap_m B_m$. By Claim 4.4, we have

$$\underline{g}(x) \geq \sup_m g(\min B_m) = \inf_m g(\max B_m) \geq \bar{g}(x).$$

Clearly there is a computable dense sequence of rationals $\{v_i\}_{i \in \mathbb{N}}$ that lists without repetition the set V of endpoints of intervals in the tree. By definition, $g(v_i)$ is a computable real uniformly in i . Since \underline{g} is continuous nondecreasing, by Proposition 2.2 we may conclude that \underline{g} is computable. This establishes the claim.

From now on we will use the letter g to denote the function extended to $[0, 1]$.

Claim 4.6. *We have $\overline{D}g(z) = \infty$.*

Let \mathcal{C} denote the tree of intervals built during the construction. Note that for each $\epsilon > 0$ there are only finitely many intervals in \mathcal{C} of length greater than ϵ . To prove the claim, we show that the strategy Γ succeeds on z in the sense that

$$\sup_{z \in A \in \mathcal{C}} \Gamma(A) = \infty.$$

By the definition of the Γ -values in (6) this will imply $\overline{D}g(z) = \infty$: let $([a_n, b_n])_{n \in \mathbb{N}}$ be a sequence of intervals containing z such that $\Gamma([a_n, b_n]) = S_g(a_n, b_n)$ is unbounded and $\lim_n (b_n - a_n) = 0$. If $z \in V$, then necessarily $a_n = z$ or $b_n = z$ for almost all n . This clearly implies $\overline{D}g(z) = \infty$. If $z \notin V$, then $a_n, b_n \neq z$ for all n , and we have $S_g(a_n, b_n) \leq \max\{S_g(a_n, z), S_g(z, b_n)\}$ by Fact 2.4. This also implies that $\overline{D}g(z) = \infty$.

To show that Γ succeeds, first we verify that Γ changes its state infinitely often on intervals B such that $z \in B$. Suppose Γ entered the betting state in (II.b). Following the notation in (II.b), let A_k be the (p, q) -interval containing z . By the first line in Lemma 4.3 and the definition of K , there is a (p, q) -interval $A \subseteq A_k$ containing z such that $S_f(A) > \gamma + 2^{-K}$. Thus $S_f(A)_K > \gamma$ and Γ will enter the non-betting state when it processes this interval, if not before.

Similarly, once Γ enters the non-betting state on an interval A_k containing z , by the second line of Lemma 4.3 it will revert to the betting state on some (r, s) -interval $B \subseteq A_k$ containing z .

Now suppose Γ enters the betting state on A , B is a largest sub-interval of A such that Γ enters the non-betting state on B , and then again, C is a largest sub-interval of B such that Γ enters the betting state on C . Then $S_f(A)_K < \beta$ while $S_f(B)_K > \gamma$, so $\Gamma(B) = \Gamma(A)S_f(B)/S_f(A) \geq \Gamma(A)^{\frac{\gamma - 2^{-K}}{\beta + 2^{-K}}} \geq \Gamma(A)\alpha$, where the last inequality follows from the definition of K in 4.1.4. Also $\Gamma(B) = \Gamma(C)$. Thus after the strategy has entered the betting state for $n + 1$ times on intervals containing z , we have $\Gamma(A) \geq \alpha^n$. Since $\alpha > 1$, this implies that Γ succeeds on z .

Remark 4.7. Given f as in Theorem 4.1, the method of the foregoing proof enables us to uniformly obtain a computable nondecreasing function g such that $f'(z) \uparrow$ implies $\overline{D}g(z) = \infty$ for all $z \in [0, 1]$. We simply sum up all the possibilities for g with appropriate scaling factors. These possibilities are f itself (for the case that already $\underline{D}f(z) = \infty$), and all the functions g obtained for any values of rationals $0 \leq \tilde{\beta} < \tilde{\gamma}$ in the construction above. We fix an effective listing $(g_k)_{k \in \mathbb{N}}$ of all such functions, and let $g = f + \sum_k 2^{-k} g_k$. Note that g is representation reducible to f : from a name for f we can compute a name for g .

4.2. Proof of Lemma 4.2. We may assume that $0 < x < y < 1/2$. Let k be an odd prime number such that $1 + 8/k < \alpha$. Let

$$\begin{aligned} P &= \{l/k: l \in \mathbb{N} \text{ \& } k/2 \leq l \leq k\}, \\ Q &= \{v/k: v \in \mathbb{Z} \text{ \& } |v| \leq k\} \end{aligned}$$

We claim that

$$L = P \cup PQ$$

is a finite set of rationals as required. Informally P is a set of scaling factors and PQ is a set of shifts.

Finding A. To obtain $A \supset [x, y]$, let $n \in \mathbb{N}$ be largest such that $y - x < (1 - 1/k)2^{-n}$, and let $\eta = 1/(2^n k)$. Informally η is the “resolution” for a discrete version of the picture that will suffice to find A and B . By the definitions we have

$$y - x + \eta < 2^{-n}. \quad (9)$$

Pick the least scaling factor $p \in P$ such that

$$y - x + \eta < p2^{-n}. \quad (10)$$

Then because of the minimality of p we have

$$p2^{-n} \leq y - x + 2\eta. \quad (11)$$

(In the case that $p = 1/2$, this follows from the maximality of n .)

Let $M \in \mathbb{N}$ be greatest such that $M\eta < x/p$. Now comes the key step: since k and 2^n are coprime, in the abelian group \mathbb{Q}/\mathbb{Z} , the elements $1/k$ and $1/2^n$ together generate the same cyclic group as η by Bézout’s Theorem. Working still in \mathbb{Q}/\mathbb{Z} , there are $i, v_0 \in \mathbb{N}$, $0 \leq i < 2^n$, $v_0 \leq k$ such that $[i/2^n] + [v_0/k] = [M\eta]$. Then, since $M\eta \leq 1$, there is an integer v , $|v| \leq k$, such that

$$i/2^n + v/k = M\eta. \quad (12)$$

To define the L -interval A , let $q = v/k \in Q$. Let

$$A = p[i2^{-n}, (i+1)2^{-n}] + pq.$$

Write $A = [a, b]$. We verify that A is as required.

Firstly, $a = pi2^{-n} + pq = pM\eta < x$, and $x - a \leq p\eta \leq \eta$ because of the maximality of M and because $p \leq 1$.

Secondly, $|A| = p2^{-n}$, so we have by (10) and (11) that $y < b < y + 2\eta$. Then

$$|A| \leq y - x + 2\eta = y - x + 2/(2^n k) \leq y - x + 8(y - x)/k,$$

where the last inequality holds because $2^{-n} \leq 4(y - x)$ by the maximality of n . Thus $|A|/(y - x) \leq 1 + 8/k < \alpha$, as required.

Finding B. Let $\alpha = 1 + 2\epsilon$. The second statement of the lemma can be derived from the first statement for the precision factor $1 + \epsilon$. Let L be the finite set of rationals obtained in the first statement for $1 + \epsilon$ in place of α . Given an interval $[x, y]$, let $[u, v] \subseteq [x, y]$ be the sub-interval such that

$$u - x = y - v = \epsilon(v - u).$$

By the first statement of the lemma there is an L -interval $B = [a, b] \supseteq [u, v]$ such that $|B|/(v - u) < 1 + \epsilon$. Then

$$\begin{aligned} u - a &< \epsilon(v - u) = u - x \\ b - v &< \epsilon(v - u) = y - v, \end{aligned}$$

whence $B \subset [x, y]$. Clearly, $(y - x)/|B| < (y - x)/(v - u) = \alpha$.

4.3. Proof of (i) \rightarrow (iii). We will extend, without mention, previous results for functions defined on $[0, 1]$ to functions defined on closed intervals with rational endpoints.

Suppose $\overline{D}g(z) = \infty$ where $g: [0, 1] \rightarrow \mathbb{R}$ is a computable nondecreasing function. We may assume that z is irrational and $1/3 \leq z \leq 2/3$.

We want to show that z is not computably random. We will apply Lemma 4.2 and its proof for $\alpha = 4$. We can choose k to be the prime 3. This yields a set L of at most 16 rationals. Since $\overline{D}g(z) = \infty$, there is $p \in \{1, \frac{2}{3}\}$ and q of the form $pv/3$, where v is an integer and $|v| \leq 3$, such that

$$\infty = \sup\{S_g(A) : A \text{ is } (p, q)\text{-interval \& } z \in A\}. \quad (13)$$

For a binary string σ let

$$A_\sigma = p[0.\sigma, 0.\sigma + 2^{-|\sigma|}] + q,$$

so that $A_\sigma \subseteq [-1, 2]$. We may assume that the given computable nondecreasing function g is actually defined on $[-1, 2]$, by letting $g(x) = g(0)$ for $x \in [-1, 0]$ and $g(x) = g(1)$ for $x \in [1, 2]$ (this extended function is computable by Proposition 2.2). We define a computable martingale N by

$$N(\sigma) = S_g(A_\sigma).$$

Now let w be the irrational number $(z - q)/p$. Then N succeeds on the binary expansion of w . For, given $c > 0$, by (13) let σ be a string such that $z \in A_\sigma$ and $S_g(A_\sigma) \geq c$. Then σ is an initial segment of the binary expansion of w and $N(\sigma) \geq c$.

By Proposition 3.4 there is a computable martingale M with the savings property that succeeds on the binary expansion of w . Let $h = \text{Fcn}(M)$, (recall that h is defined on $[0, 1]$). By the implication (i) \rightarrow (ii) of Theorem 3.6 we have $\underline{D}h(w) = \infty$.

To show that z is not computably random, we define a nondecreasing computable function r . Let $d = h((1/3 - q)/p)$. Let

$$r(x) = h((x - q)/p) - d \text{ for } x \in [1/3, 2/3].$$

Furthermore let $r(x) = 0$ for $0 \leq x < 1/3$, and $r(x) = r(2/3)$ for $2/3 < x \leq 1$. Clearly the function r is computable and satisfies $r(0) = 0$, so if we let V be the computable martingale $\text{Mart}(r)$, then we have $\text{Fcn}(V) = r$ by Fact 3.5. Since we assumed that $z \in [1/3, 2/3]$, we have $\underline{D}r(z) = \underline{D}h(w)/p = \infty$. By the implication (ii) \rightarrow (i) of Theorem 3.6 (which does not rely on the savings property), we may conclude that V succeeds on z . \square

4.4. Remarks and applications. In this subsection we make some remarks on Theorem 4.1 and provide some of its consequences. We include an application to \mathcal{L}_1 -computable functions.

A consequence of the proof of Theorem 4.1. Recall names for nondecreasing functions from Subsection 2.3.

Corollary 4.8. *There is a procedure to compute from a name of a nondecreasing function f the binary expansion of a real z such that $f'(z)$ exists. In particular, each computable nondecreasing function is differentiable at a computable real.*

Proof. From a name for f we can compute a name for a nondecreasing function g as in Remark 4.7 above. Let V be the martingale obtained in the proof of (i) \rightarrow (iii) above. Note that V , viewed as a function from binary strings to Cauchy names for reals, is computable in a name for g , and hence in a name for f .

For any $z \in [1/3, 2/3]$, we have

$$f'(z) \uparrow \Rightarrow \overline{D}g(z) = \infty \Rightarrow V \text{ succeeds on the binary expansion of } z.$$

It remains to compute from a name for V the binary expansion Z of a real $z \in [1/3, 2/3]$ such that $V(Z \upharpoonright_n)$ is bounded. Let the first 3 bits of Z be 1, 0, 0. For $n \geq 3$, if $\sigma = Z \upharpoonright_n$ has been determined, use V to determine a bit $Z(n) = b$ such that $V(\sigma \hat{\ } b) \leq V(\sigma \hat{\ } (1 - b)) + 2^{-n}$. Clearly, $\sup_n V(Z \upharpoonright_n) < \infty$. \square

Note that even if f is computable, the uniformity in the previous result is only on names. From each name for f we can uniformly compute (the binary expansion) of a real z such that $f'(z) \downarrow$, but different names for f might result in different reals.

Some new facts on computable randomness. As a consequence of Theorem 4.1 we obtain a preservation result for computable randomness similar to Corollary 5.8.

Corollary 4.9. *Suppose $z \in \mathbb{R}$ is computably random. Let H be a computable function that is monotonic and 1-1 in a neighborhood of z . If $H'(z) \neq 0$, then $H(z)$ is computably random.*

Proof. First suppose H is increasing in a neighborhood of z . If a function f is computable and nondecreasing in a neighborhood of $H(z)$, then the composition $f \circ H$ is nondecreasing in a neighborhood of z . Thus, since z is computably random, $(f \circ H)'(z)$ exists. Since $H'(z) \neq 0$ and H is continuous and 1-1 in a neighborhood of z , this implies that $f'(H(z))$ exists. Hence $H(z)$ is computably random.

If H is decreasing, we apply the foregoing argument to $-H$ instead. \square

Next, we consider differentiability of computable Lipschitz functions.

Corollary 4.10. *If a real $z \in [0, 1]$ is computably random, then each computable Lipschitz function on the unit interval is differentiable at z .*

Proof. Suppose h is Lipschitz via a constant $C \in \mathbb{N}$. Then the function f given by $f(x) = Cx - h(x)$ is computable and nondecreasing. Thus, by (i) \rightarrow (ii) of Theorem 4.1, f and hence h is differentiable at z . \square

The converse of Cor. 4.10 has been shown in [11]. Note that the function f in the foregoing proof is Lipschitz. Thus, computable randomness is characterized by differentiability of computable functions that are monotonic, or Lipschitz, or both monotonic and Lipschitz. This accounts for some arrows in Figure 1 in the introduction.

\mathcal{L}_1 -computable functions. Recall that $\mathcal{L}_1([a, b])$ denotes the set of integrable functions $g : [a, b] \rightarrow \mathbb{R}$, and $\|g\|_1 = \int_{[a, b]} |g| d\lambda$. As in [18, p. 84] we say that $g : [0, 1] \rightarrow \mathbb{R}$ is \mathcal{L}_1 -computable if there is a uniformly computable sequence $(h_n)_{n \in \mathbb{N}}$ of functions on $[0, 1]$ such that $\|g - h_n\|_1 \leq 2^{-n}$ for each n . (The notion of computability for the h_n is the usual one in the sense of Subsection 2.2; in particular, they are continuous.)

Recall that for a function h , we let $h^+ = \max(h, 0)$ and $h^- = \max(-h, 0)$. If g is \mathcal{L}_1 -computable via $(h_n)_{n \in \mathbb{N}}$, then g^+ is \mathcal{L}_1 -computable via $(h_n^+)_{n \in \mathbb{N}}$, and g^- is \mathcal{L}_1 -computable via $(h_n^-)_{n \in \mathbb{N}}$. (This follows because $|g^+(x) - h_n^+(x)| \leq |g(x) - h_n(x)|$, etc.)

If $g \in \mathcal{L}_1([0, 1])$ then its restriction to the interval $[0, x]$ is in $\mathcal{L}_1([0, x])$. Let $G(x) = \int_0^x g d\lambda$.

Fact 4.11. *If g is \mathcal{L}_1 -computable then G is computable.*

Proof. We have $G = G^+ - G^-$ where $G^+(x) = \int_0^x g^+ d\lambda$ and $G^-(x) = \int_0^x g^- d\lambda$. For each \mathcal{L}_1 -computable function f , $\int_0^q f d\lambda$ is computable uniformly in a rational q by [17, Lemma 2.3]. Thus the nondecreasing continuous functions G^+, G^- are computable by Proposition 2.2. \square

The Lebesgue differentiation theorem 2.6 was discussed in Subsection 2.7: if g is in $\mathcal{L}_1([0, 1])$ and G is as above, then for λ -almost every z , $G'(z)$ exists and equals $g(z)$. For the mere existence of $G'(z)$, we have the following corollary to Theorem 4.1.

Corollary 4.12. *Let g be \mathcal{L}_1 -computable. Then $G'(z)$ exists for each computably random real z .*

Proof. It suffices to observe that $(G^+)'(z)$ and $(G^-)'(z)$ exist. \square

Note that the integrable function g from the proof of Claim 5.6 is not \mathcal{L}_1 -computable, even though its indefinite integral is computable: if g were \mathcal{L}_1 -computable, then the function f from Lemma 5.5 would be differentiable at each computably random real.

In fact, the condition in Cor. 4.12 that $G'(z)$ exists for each \mathcal{L}_1 -computable function g characterizes Schnorr randomness by results of Rute, and Pathak, Rojas and Simpson, as mentioned in the introduction.

5. WEAK 2-RANDOMNESS AND MARTIN-LÖF RANDOMNESS

In this section, when discussing inclusion, disjointness, etc., for open sets in the unit interval, we will ignore the elements that are dyadic rationals. For instance, we view the interval $(1/4, 3/4)$ as the union of $(1/4, 1/2)$ and $(1/2, 3/4)$. With this convention, the clopen sets in Cantor space 2^ω correspond to the finite unions of open intervals with dyadic rational endpoints.

5.1. Characterizing weak 2-randomness in terms of differentiability. Recall that a real z is *weakly 2-random* if z is in no null Π_2^0 set.

Theorem 5.1. *Let $z \in [0, 1]$. Then*

z is weakly 2-random \Leftrightarrow

each a.e. differentiable computable function is differentiable at z .

Proof. “ \Rightarrow ”: For rationals p, q let

$$\begin{aligned}\underline{C}(p) &= \{z: \forall t > 0 \exists h [0 < |h| \leq t \ \& \ S_f(z, z+h) < p]\} \\ \overline{C}(q) &= \{z: \forall t > 0 \exists h [0 < |h| \leq t \ \& \ S_f(z, z+h) > q]\},\end{aligned}$$

where t, h range over rationals. The function $z \mapsto S_f(z, z+h)$ is computable, and its index in the sense of Subsection 2.2 can be obtained uniformly in h . Hence the set

$$\{z: S_f(z, z+h) < p\}$$

is a Σ_1^0 set uniformly in p, h by Lemma 2.3 and its uniformity in the strong form remarked after its proof. Thus $\underline{C}(p)$ is a Π_2^0 set uniformly in p . Similarly, $\overline{C}(q)$ is a Π_2^0 set uniformly in q . Clearly,

$$\underline{D}f(z) < p \Rightarrow z \in \underline{C}(p) \Rightarrow \underline{D}f(z) \leq p,$$

$$\overline{D}f(z) > q \Rightarrow z \in \overline{C}(q) \Rightarrow \overline{D}f(z) \geq q.$$

Therefore $f'(z)$ fails to exist iff

$$\forall p [z \in \underline{C}(p)] \vee \forall q [z \in \overline{C}(q)] \vee \exists p \exists q [p < q \ \& \ z \in \underline{C}(p) \ \& \ z \in \overline{C}(q)],$$

where p, q range over rationals. This shows that $\{z: f'(z) \text{ fails to exist}\}$ is a Σ_3^0 set (i.e., an effective union of Π_2^0 sets). If f is a.e. differentiable then this set is null and thus cannot contain a weakly 2-random.

“ \Leftarrow ”: For an interval $A \subseteq [0, 1]$ and $p \in \mathbb{N}$ let $\Lambda_{A,p}$ be the “sawtooth function” that is constant 0 outside A , reaches $p|A|/2$ at the middle point of A and is linearly interpolated elsewhere. Thus $\Lambda_{A,p}$ has slope $\pm p$ between pairs of points in the same half of A , and

$$\Lambda_{A,p}(x) \leq p|A|/2, \tag{14}$$

for each x .

Let $(\mathcal{G}_m)_{m \in \mathbb{N}}$ be a sequence of uniformly Σ_1^0 sets in the sense of Subsection 2.4, where $\mathcal{G}_m \subseteq [0, 1]$, such that $\mathcal{G}_m \supseteq \mathcal{G}_{m+1}$ for each m . We build a computable function f such that $f'(z)$ fails to exist for every $z \in \bigcap_m \mathcal{G}_m$. To establish the implication \Leftarrow , we also show in Claim 5.4 that the function f is a.e. differentiable in the case that $\bigcap_m \mathcal{G}_m$ is null.

Recall the convention that we ignore the dyadic rationals when discussing inclusion, union, disjointness, etc., for open sets in the unit interval. We have an effective enumeration $(D_{m,l})_{m,l \in \mathbb{N}}$ of open intervals with dyadic rational endpoints such that

$$\mathcal{G}_m = \bigsqcup_{l \in \mathbb{N}} D_{m,l},$$

for each m . We may assume, without loss of generality, that for each $D_{m+1,k}$, there is an l such that $D_{m+1,k} \subseteq D_{m,l}$.

We construct by recursion on m a computable double sequence $(C_{m,i})_{m,i \in \mathbb{N}}$ of open intervals with dyadic rational endpoints such that $\bigsqcup_i C_{m,i} = \mathcal{G}_m$,

$$C_{m,i} \cap C_{m,k} = \emptyset \text{ and } |C_{m,i}| \geq |C_{m,k}| \text{ for } i < k, \quad (15)$$

and, furthermore, if $B = C_{m,i}$ for $m > 0$, then there is an interval $A = C_{m-1,k}$ such that

$$B \subseteq A \text{ \& } |B| \leq 8^{-m}|A|. \quad (16)$$

Each interval of the form $D_{m,k}$ will be a finite union of intervals of the form $C_{m,i}$.

Construction of the double sequence $(C_{m,i})_{m,i \in \mathbb{N}}$.

Suppose $m = 0$, or $m > 0$ and we have already defined $(C_{m-1,j})_{j \in \mathbb{N}}$. Define $(C_{m,i})_{i \in \mathbb{N}}$ as follows.

Suppose $N \in \mathbb{N}$ is greatest such that we have already defined $C_{m,i}$ for $i < N$. When a new interval $D = D_{m,l}$ with dyadic rational endpoints is enumerated into \mathcal{G}_m , if $m > 0$ we wait until D is contained in a union of intervals $\bigcup_{r \in F} C_{m-1,r}$, where F is finite. This is possible because D is contained in a single interval in \mathcal{G}_{m-1} , and this single interval was handled in the previous stage of the recursion. If $m > 0$, let δ be the minimum of $|D|$ and the lengths of these finitely many intervals; if $m = 0$, let $\delta = |D|$. Let ϵ be the minimum of $|C_{m,N-1}|$ (if $N > 0$), and $8^{-m}2^{-l}\delta$. (We will need the factor 2^{-l} when we show in Claim 5.4 that f is a.e. differentiable.)

We partition D into disjoint sub-intervals $C_{m,i}$ with dyadic rational endpoints, $i = N, \dots, N' - 1$, and of nonincreasing length at most ϵ , so that in case $m > 0$ each of the sub-intervals is contained in an interval A of the form $C_{m-1,r}$ for some $r \in F$.

Now let

$$\begin{aligned} f_m &= \sum_{i=0}^{\infty} \Lambda_{C_{m,i}, 4^m}(m \in \mathbb{N}) \\ f &= \sum_{m=0}^{\infty} f_m. \end{aligned}$$

Since $|C_{m,i}| \leq 8^{-m}$ for each i , we have $f_m(x) \leq 8^{-m}4^m/2 \leq 2^{-m-1}$ for each x .

Claim 5.2. *The function f is computable.*

Since $f_m(x) \leq 2^{-m-1}$ for each m , $f(x)$ is defined for each $x \in [0, 1]$. We first show that $f(q)$ is computable uniformly in a rational q . Given $m > 0$, since $|C_{m,i}| \rightarrow_i 0$, we can find i^* such that

$$|C_{k,i^*}| \leq 8^{-m}/(m+1) \text{ for each } k \leq m.$$

Then, since the length of the intervals $C_{k,i}$ is nonincreasing in i and by (14), we have $\Lambda_{C_{k,i}, 4^k}(q) \leq 2^{-m-1}/(m+1)$ for all $k \leq m$ and $i \geq i^*$. So by the disjointness in (15), $\sum_{k \leq m} \sum_{i \geq i^*} \Lambda_{C_{k,i}, 4^k}(q) \leq 2^{-m-1}$. We also have $\sum_{k > m} f_k(q) \leq \sum_{k > m} 2^{-k-1} = 2^{-m-1}$. Hence the approximation to $f(q)$ at

stage i^* based only on the intervals of the form $C_{k,i}$ for $k \leq m$ and $i < i^*$ is within 2^{-m} of $f(q)$.

To show f is computable, by Subsection 2.2 it suffices now to verify that f is effectively uniformly continuous. Suppose $|x - y| \leq 8^{-m}$. For $k < m$, we have $|f_k(x) - f_k(y)| \leq 4^k|x - y|$. For $k \geq m$ we have $f_k(x), f_k(y) \leq 2^{-k-1}$. Thus

$$|f(x) - f(y)| \leq |x - y| \sum_{k < m} 4^k + \sum_{k \geq m} 2^{-k} < 2^{-m+2}.$$

Claim 5.3. *Suppose $z \in \bigcap \mathcal{G}_m$. Then $\overline{D}f(z) = \infty$ or $\underline{D}f(z) = -\infty$.*

For each m there is an interval A_m of the form $C_{m,i}$ such that $z \in A_m$. Suppose first that there are infinitely many m such that z is in the left half of A_m . We show $\overline{D}f(z) = \infty$. Let m be one such value. Choose

$$h = \pm|A_m|/4$$

so that $z + h$ is also in the left half of A_m . We show that the slope

$$S_{f_m}(z, z + h) = 4^m$$

does not cancel out with the slopes, possibly negative, that are due to other f_k . If $k < m$ then we have $|S_{f_k}(z, z + h)| \leq 4^k$. Suppose $k > m$. Then by (14) and (16) we have $f_k(x) \leq 4^k 8^{-k}|A_m|/2 = 2^{-k-1}|A_m|$ for $x \in \{z, z + h\}$ and hence

$$|S_{f_k}(z, z + h)| \leq \frac{2^{-k}|A_m|}{|h|} = 2^{-k+2}.$$

Therefore, for $m > 0$

$$S_f(z, z + h) \geq 4^m - \sum_{k < m} 4^k - \sum_{k > m} 2^{-k+2} \geq 4^{m-1} - 4.$$

Thus $\overline{D}f(z) = \infty$.

If there are infinitely many m such that z is in the right half of A_m , then $\underline{D}f(z) = -\infty$ by a similar argument.

Claim 5.4. *If $\bigcap_m \mathcal{G}_m$ is null, then f is differentiable almost everywhere.*

Let $\widehat{D}_{m,l}$ be the open interval in \mathbb{R} with the same middle point as $D_{m,l}$ such that $|\widehat{D}_{m,l}| = 3|D_{m,l}|$. Let $\widehat{\mathcal{G}}_m = [0, 1] \cap \bigcup_l \widehat{D}_{m,l}$. Clearly $\lambda \widehat{\mathcal{G}}_m \leq 3\lambda \mathcal{G}_m$, so that $\bigcap_m \widehat{\mathcal{G}}_m$ is null.

We show that $f'(z)$ exists for each $z \notin \bigcap_m \widehat{\mathcal{G}}_m$ that is not a dyadic rational. In the following, let h, h_0 , etc., range over rationals. Note that

$$S_f(z, z + h) = \sum_{k=0}^{\infty} S_{f_k}(z, z + h).$$

Let m be the least number such that $z \notin \widehat{\mathcal{G}}_m$. Since z is not a dyadic rational, we may choose $h_0 > 0$ such that for each $k < m$, the function f_k is linear in the interval $[z - h_0, z + h_0]$. So for $|h| \leq h_0$ the contribution of these f_k to the slope $S_f(z, z + h)$ is constant. It now suffices to show that

$$\lim_{h \rightarrow 0} \sum_{r=m}^{\infty} |S_{f_r}(z, z + h)| = 0.$$

Note that f_r is nonnegative and $f_r(z) = 0$ for $r \geq m$. Thus it suffices, given $\epsilon > 0$, to find a positive $h_1 \leq h_0$ such that

$$\sum_{r=m}^{\infty} f_r(z+h) \leq \epsilon|h| \quad (17)$$

whenever $|h| \leq h_1$.

Roughly, the idea is the following: take $r \geq m$. If $f_r(z+h) \neq 0$ then $z+h$ is in some $D_{m,l}$. Because $z \notin \widehat{D}_{m,l}$, $|h| \geq |D_{m,l}|$. We make sure that $f_r(z+h)$ is small compared to $|h|$ by using that the height of the relevant sawtooth depends on the length of its base interval $C_{r,v}$ containing $z+h$, and that this length is small compared to h .

We now provide the details on how to find h_1 as above. Choose $l^* \in \mathbb{N}$ such that $2^{-l^*} \leq \epsilon$. If $C_{m,i} \subseteq D_{m,l}$ and $l \geq l^*$, we have

$$|C_{m,i}| \leq 8^{-m} \epsilon |D_{m,l}|. \quad (18)$$

Let $h_1 = \min\{|D_{m,l}| : l < l^*\}$. Suppose $h > 0$ and $|h| \leq h_1$.

Firstly we consider the contribution of f_m to (17). If $f_m(z+h) > 0$ then $z+h \in C_{m,i} \subseteq D_{m,l}$ for some (unique) l, i . Since $z \notin \widehat{D}_{m,l}$ and $|h| \leq h_1$, we have $|h| \geq |D_{m,l}|$ and $l \geq l^*$. By (14), (18) and the definition of f_m ,

$$f_m(z+h) \leq 4^m |C_{m,i}|/2 \leq 2^{-m-1} |D_{m,l}| \epsilon.$$

Thus $f_m(z+h) \leq 2^{-m-1} \epsilon |h|$.

Next, we consider the contribution of f_r , $r > m$, to (17). If $f_r(z+h) > 0$ then $z+h \in C_{r,v} \subseteq C_{m,i}$ for some v . Thus, by construction,

$$f_r(z+h) \leq 4^r |C_{r,v}|/2 \leq 4^r 8^{-r} |C_{m,i}|/2 \leq 2^{-r-1} |D_{m,l}| \epsilon \leq 2^{-r-1} \epsilon |h|.$$

This establishes (17) and completes the proof. \square

5.2. Characterizing Martin-Löf randomness in terms of differentiability. Recall that a function $f: [0, 1] \rightarrow \mathbb{R}$ is of *bounded variation* if

$$\infty > \sup \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|,$$

where the sup is taken over all collections $t_1 < t_2 < \dots < t_n$ in $[0, 1]$. A stronger condition on f is *absolute continuity*: for every $\epsilon > 0$, there is $\delta > 0$ such that

$$\epsilon > \sup \sum_{i=1}^n |f(b_i) - f(a_i)|,$$

for every collection $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \leq 1$ such that $\delta > \sum_{i=1}^n b_i - a_i$. The absolutely continuous functions are precisely the indefinite integrals of functions in $\mathcal{L}_1([0, 1])$ (see [1, Thm. 5.3.6]). Note that it is easy to construct a computable differentiable function that is not of bounded variation.

We will characterize Martin-Löf randomness via differentiability of computable functions of bounded variation, following the scheme (*) in the introduction. For the implication \Leftarrow , an appropriate single function suffices, because there is a universal Martin-Löf test.

Lemma 5.5 ([6], Example 2). *There is a computable function f of bounded variation (in fact, absolutely continuous) such that $f'(z)$ exists only for Martin-Löf random reals z .*

Proof. Let $(\mathcal{G}_m)_{m \in \mathbb{N}}$ be a universal Martin-Löf test, where $\mathcal{G}_m \subseteq [0, 1]$, such that $\mathcal{G}_m \supseteq \mathcal{G}_{m+1}$ for each m . We may assume that $\lambda \mathcal{G}_m \leq 8^{-m}$. Define a computable function f as in the proof of the implication \Leftarrow of Theorem 5.1. By Claim 5.3, $f'(z)$ fails to exist for any $z \in \bigcap_m \mathcal{G}_m$, i.e., for any z that is not Martin-Löf random. It remains to show the following.

Claim 5.6. *f is absolutely continuous, and hence of bounded variation.*

For an open interval $A \subseteq [0, 1]$, let $\Theta_{A,p}$ be the function that is undefined at the endpoints and the middle point of A , has value p on the left half, value $-p$ on the right half of A , and is 0 outside A . Then $\int_0^x \Theta_{A,p} = \Lambda_{A,p}(x)$.

Let $g_m = \sum_i \Theta_{C_{m,i}, 4^{-m}}$. Note that $\lambda \mathcal{G}_m \leq 8^{-m}$ implies that g_m is integrable with $\int |g_m| \leq 2^{-m}$, and hence $\sum_m \int |g_m| \leq 2$. Then, by a well-known corollary to the Lebesgue dominated convergence theorem (see for instance [19, Thm. 1.38]), the function $g(y) = \sum_m g_m(y)$ is defined a.e., g is integrable, and $\int_0^x g = \sum_m \int_0^x g_m$. Since $f_m(x) = \int_0^x g_m$, this implies that $f(x) = \int_0^x g$. Thus, f is absolutely continuous. \square

We now arrive at the analytic characterization of Martin-Löf randomness originally due to Demuth [6]. The implication (i) \rightarrow (ii) below restates [6, Thm. 3] in classical language.

Theorem 5.7. *The following are equivalent for $z \in [0, 1]$:*

- (i) *z is Martin-Löf random.*
- (ii) *Every computable function f of bounded variation is differentiable at z .*
- (iii) *Every computable function f that is absolutely continuous is differentiable at z .*

Proof. The implication (iii) \rightarrow (i) follows from Lemma 5.5. The implication (ii) \rightarrow (iii) follows because each absolutely continuous function has bounded variation. Our proof of implication (i) \rightarrow (ii) relies on a result on computable randomness below, and is therefore postponed to Subsection 6.2. \square

We say that a real $z \in \mathbb{R}$ satisfies an algorithmic randomness notion iff its fractional part $z - \lfloor z \rfloor$ satisfies it. As a corollary we obtain a preservation result for Martin-Löf randomness. For instance, if $z > 0$ is Martin-Löf random then so are z^4 , \sqrt{z} , and $1/z$. More generally, for each computable real $\alpha \neq 0$, Martin-Löf randomness is preserved under the map $z \rightarrow z^\alpha$.

Corollary 5.8. *Suppose $z \in \mathbb{R}$ is Martin-Löf random. Let H be a computable function that is Lipschitz and 1-1 in a neighborhood of z . If $H'(z) \neq 0$, then $H(z)$ is Martin-Löf random.*

Proof. Let f be an arbitrary function that is computable and absolutely continuous in a neighborhood of $H(z)$. Then the composition $f \circ H$ is absolutely continuous in a neighborhood of z . Thus, since z is Martin-Löf random, $(f \circ H)'(z)$ exists. Since H is continuous and 1-1 in a neighborhood of z , $H'(z) \neq 0$ implies that $f'(H(z))$ exists. Hence $H(z)$ is Martin-Löf random by Theorem 5.7. \square

6. EXTENSIONS OF THE RESULTS

So far, we have proved two instances of equivalences of type $(*)$ at the beginning of the paper: for weak 2-randomness, and for computable randomness. We have also stated a result for Martin-Löf randomness in Theorem 5.7, but not yet proved the implication \Rightarrow in $(*)$. We will see that these equivalences do not rely on the full hypothesis that the functions in the relevant class are computable.

Computability on $I_{\mathbb{Q}}$. Recall that $I_{\mathbb{Q}} = [0, 1] \cap \mathbb{Q}$. We say that a function f is *computable on $I_{\mathbb{Q}}$* if its domain contains $I_{\mathbb{Q}}$, and $f(q)$ is a computable real uniformly in $q \in I_{\mathbb{Q}}$. Even if a function f that is computable on $I_{\mathbb{Q}}$ has domain $[0, 1]$, it need not be continuous: for instance, let $f(x) = 0$ for $x^2 \leq 1/2$, and $f(x) = 1$ for $x^2 > 1/2$. In fact, computability of a function f on $I_{\mathbb{Q}}$ is so general that it can barely be considered a genuine notion from computable analysis: we merely require that $(f(q))_{q \in I_{\mathbb{Q}}}$ can be viewed as a computable array of reals indexed by the rationals in $[0, 1]$, similar to computable sequences of reals in the sense of Subsection 2.1.

Nonetheless, in this section we will show that this much weaker effectivity hypothesis is sufficient for the implications \Rightarrow in $(*)$, including the case of Martin-Löf randomness. Of course, if f is not defined in a whole neighborhood of a real z , we lose the usual notion of differentiability at z . Instead, we will consider pseudo-differentiability at z , where one only looks at the slopes at smaller and smaller intervals containing z that have rational endpoints. If f is total and continuous (e.g., if f is computable), then pseudo-differentiability coincides with usual differentiability, as we will see in Fact 6.2. Thus, the result for Martin-Löf randomness also supplies our proof of the implication (i) \rightarrow (ii) of Theorem 5.7, which we had postponed to this section.

The implications \Leftarrow in previous proofs of results of type $(*)$ always produce a *computable* function f such that $f'(z)$ fails to exist if the real z is not random in the appropriate sense. Since computability implies being computable on $I_{\mathbb{Q}}$, we now have full equivalences of type $(*)$ where the effectivity notion is computability on $I_{\mathbb{Q}}$.

Markov computable functions. The extensions of our results are interesting because a number of effectivity notions for functions have been studied in computable analysis that are intermediate between being computable and computable on $I_{\mathbb{Q}}$. Hence we also obtain equivalences of type $(*)$ for these effectivity notions.

We briefly introduce Markov computability here. See [2, 21] for more detail, and for other intermediate effectivity notions for functions such as the slightly weaker Mazur computability (where computable sequences of reals are mapped to computable sequences of reals).

Recall from Subsection 2.2 that a Cauchy name for a real x is a sequence $L = (q_n)_{n \in \mathbb{N}}$ (i.e., a function $L: \mathbb{N} \rightarrow \mathbb{Q}$) such that $|q_n - q_{n-1}| \leq 2^{-n}$ for $n > 0$. A real-valued function f defined on all computable reals in $[0, 1]$ is called *Markov computable* if there is a computable function $h: \mathbb{N} \rightarrow \mathbb{N}$ such that, if ϕ_e is a Cauchy name of x , then $\phi_{h(e)}$ is a Cauchy name of $f(x)$. This notion has been studied in the Russian school of constructive analysis, e.g.,

by Ceitin [4], and later by Demuth [7], who used the term “constructive function”.

Clearly, being Markov computable implies computability on $I_{\mathbb{Q}}$. But Markov computability is much stronger. For instance, each Markov computable function is continuous on the computable reals. In particular, the $I_{\mathbb{Q}}$ -computable function given above is not Markov computable.

To see continuity, we discuss an apparently stronger notion of effectivity which is in fact equivalent to Markov computability. Recall that in Subsection 2.2 we defined a function $f: [0, 1] \rightarrow \mathbb{R}$ to be computable if there is a Turing functional Φ that maps a Cauchy name q of $x \in [0, 1]$ to a Cauchy name of $f(x)$. Suppose now $f(x)$ is at least defined for all computable reals x in $[0, 1]$. Let us restrict the definition above to computable reals: there is a Turing functional Φ such that Φ^Q is total for all *computable* Cauchy names Q , and Φ maps every computable Cauchy name for a real x to a Cauchy name for $f(x)$. Let us temporarily call such a function weakly computable. Note that a weakly computable function is continuous on the computable reals, because of the use principle: to compute $\Phi^Q(n)$, the approximation of $f(x)$ at distance at most 2^{-n} , we can use only finitely many terms of the Cauchy name Q for x .

Clearly, every weakly computable function is Markov computable. The converse implication follows from the Kreisel-Lacombe-Shoenfield/Ceitin theorem; see Moschovakis [14, Thm. 4.1] for a recent account.

6.1. Pseudo-differentiability. Recall the notations $D^V f(x)$ and $D_V(x)$ from Subsection 2.5, where $V \subseteq \mathbb{R}$ and the domain of the function f contains $V \cap [0, 1]$. We will write $\underline{D}f(x)$ for $D_{\mathbb{Q}}f(x)$, and $\tilde{D}f(x)$ for $D^{\mathbb{Q}}f(x)$.

Definition 6.1. We say that a function f with domain containing $I_{\mathbb{Q}}$ is *pseudo-differentiable at x* if $-\infty < \underline{D}f(x) = \tilde{D}f(x) < \infty$.

Fact 6.2. Suppose that $f: [0, 1] \rightarrow \mathbb{R}$ is continuous. Then

$$\underline{D}f(x) = \underline{D}f(x) \text{ and } \tilde{D}f(x) = \overline{D}f(x)$$

for each x . Thus, if f is pseudo-differentiable at x , then $f'(x) = \underline{D}f(x) = \tilde{D}f(x)$.

Proof. Fix $h > 0$. Since the slope S_f is continuous on its domain,

$$\begin{aligned} \inf\{S_f(a, b) : a, b \in I_{\mathbb{Q}} \ \& \ a \leq x \leq b \ \& \ 0 < b - a \leq h\} \\ &\leq \inf\{S_f(x, x+l) : |l| \leq h\}, \end{aligned}$$

which implies that $\underline{D}f(x) \leq \underline{D}f(x)$. The converse inequality is always true by the remarks at the end of Subsection 2.5. In a similar way, one shows that $\tilde{D}f(x) = \overline{D}f(x)$. \square

6.2. Extension of the results to the setting of computability on $I_{\mathbb{Q}}$. We will prove the implications \Rightarrow in our three results of type (*) for functions that are merely computable on $I_{\mathbb{Q}}$.

Extending the definition in Subsection 5.2, we say that a function f with domain contained in $[0, 1]$ is of bounded variation if $\infty > \sup \sum_{i=1}^n |f(t_{i+1}) - f(t_i)|$ where the sup is taken over all collections $t_1 < t_2 < \dots < t_n$ in the domain of f .

Theorem 6.3. *Let f be computable on $I_{\mathbb{Q}}$.*

- (I) *If f is nondecreasing on $I_{\mathbb{Q}}$, then f is pseudo-differentiable at each computably random real z .*
- (II) *If $f \upharpoonright I_{\mathbb{Q}}$ is of bounded variation, then f is pseudo-differentiable at each Martin-Löf random real z .*
- (III) *If f is pseudo-differentiable at almost every $x \in [0, 1]$, then f is pseudo-differentiable at each weakly 2-random real z .*

Proof. (I) We will show that the analogs of the implications (i) \rightarrow (iii) \rightarrow (ii) in Theorem 4.1 are valid when $\overline{D}g(z)$ is replaced by $\tilde{D}g(z)$, and differentiability by pseudo-differentiability.

As before, the analog of (i) \rightarrow (iii) is proved by contraposition: if g is nondecreasing and computable on $I_{\mathbb{Q}}$, and $\tilde{D}g(z) = \infty$, then z is not computably random. Note that (13) in the proof of (i) \rightarrow (iii) is still valid under the hypothesis that $\tilde{D}g(z) = \infty$. The martingale N defined there is computable under the present, weaker hypothesis that g is computable on $I_{\mathbb{Q}}$. Now, as before, we may obtain a computable martingale V that succeeds on z .

For the analog of implication (iii) \rightarrow (ii) in Theorem 4.1, we are given a function f that is nondecreasing and computable on $I_{\mathbb{Q}}$, and not pseudo-differentiable at z . We want to build a function g that is nondecreasing and computable on $I_{\mathbb{Q}}$, such that $\tilde{D}g(z) = \infty$.

If $\underline{D}f(z) = \infty$ we let $g = f$. Now suppose otherwise. We will show that Lemma 4.3 is still valid for appropriate $\beta < \gamma$. Firstly, we modify Lemma 2.5. For $h > 0$ we let

$$\mathcal{K}_h = \{\langle a, b \rangle : 0 < b - a < h \text{ \& } a + (b - a)/4 < z < b - (b - a)/4\}.$$

Lemma 6.4. *Suppose that $\lim_{h \rightarrow 0} \{S_f(a, b) : a, b \in I_{\mathbb{Q}} \text{ \& } (a, b) \in \mathcal{K}_h\}$ exists. Then f is pseudo-differentiable at z .*

To see this, take $h > 0$ and $t < s$ such that $t < S_f(a, b) < s$ for all $(a, b) \in \mathcal{K}_h$. We will use the notation from the proof of Lemma 2.5. In particular, we consider an interval (c, d) with rational endpoints containing z such that $d - c < h/3$, and, as before, define N, a_i, b_i ($0 \leq i \leq N + 1$) so that the intervals (a_i, b_i) and (a_{i+1}, b_i) ($0 \leq i \leq N$) contain z in their middle thirds.

Note that \mathcal{K}_h is open in \mathbb{R}^2 . Therefore

$$\mathcal{S} = \{\langle u_0, v_0, \dots, u_N, v_N, u_{N+1} \rangle : \forall i \leq N [\langle u_i, v_i \rangle \in \mathcal{K}_h \text{ \& } \langle u_{i+1}, v_i \rangle \in \mathcal{K}_h]\}$$

is an open subset of \mathbb{R}^{2N+3} containing $\langle a_0, b_0, \dots, b_N, a_{N+1} \rangle$.

If $\Gamma = \langle u_0, v_0, \dots, u_N, v_N, u_{N+1} \rangle$ is in $\mathcal{K}_h \cap \mathbb{Q}^{2N+3}$ then we have inequality (3) in the proof of Lemma 2.5 with u_i, v_i instead of a_i, b_i . Letting Γ tend to $\langle a_0, b_0, \dots, b_N, a_{N+1} \rangle$ we see that this implies the inequality (3) as stated, except that we now may have equality. Nonetheless may continue the argument as before in order to show $S_f(c, d) < 5s - 4t$.

The lower bound $5t - 4s < S_f(c, d)$ is proved in a similar way. This yields the lemma.

By our hypothesis that f is not pseudo-differentiable at z , the limit in the lemma does not exist, so we can choose $\tilde{\beta}, \tilde{\gamma}$ such that

$$\begin{aligned}\tilde{\gamma} &< \limsup_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \text{ \& } z \in \text{center region of } (x, y)\}, \\ \tilde{\beta} &> \liminf_{h \rightarrow 0} \{S_f(x, y) : 0 \leq y - x \leq h \text{ \& } z \in \text{center region of } (x, y)\},\end{aligned}$$

where x, y range over rationals. Choosing $\alpha < 4/3$ as before, the proof of Lemma 4.3 now goes through.

Note that the construction in the proof of (iii) \rightarrow (ii) actually yields a *computable* nondecreasing g . For, to define g we only needed to compute the values of f on the dense set $V \subseteq I_{\mathbb{Q}}$; we did not require f to be continuous.

We have $\overline{D}g(z) = \infty$ as in Claim 4.6. Since g is continuous, by Fact 6.2 this implies $\tilde{D}g(z) = \infty$.

(II) Recall names for nondecreasing function from the end of Subsection 2.3. It is not hard to show that the names of nondecreasing functions form a Π_1^0 class. Note that h has a computable name iff h is computable on the rationals.

Let the variable q range over $I_{\mathbb{Q}}$. Jordan's Theorem also holds for functions defined on $I_{\mathbb{Q}}$: if f has bounded variation on $I_{\mathbb{Q}}$ then $f \upharpoonright I_{\mathbb{Q}} = f_0 - f_1$ for nondecreasing functions f_0, f_1 defined on $I_{\mathbb{Q}}$. One simply lets $f_0(q)$ be the variation of f restricted to $[0, q] \cap I_{\mathbb{Q}}$. Then f_0 is nondecreasing. One checks as in the usual proof of Jordan's theorem (e.g., [1, Cor 5.2.3]) that the function f_1 given by $f_1(q) = f_0(q) - f(q)$ is nondecreasing as well.

Now let \mathcal{P} be the nonempty class of pairs $\langle \eta_0, \eta_1 \rangle$ of names for pairs f_0, f_1 of nondecreasing functions on $I_{\mathbb{Q}}$ such that $f(q) = f_0(q) - f_1(q)$ for each $q \in I_{\mathbb{Q}}$. Since f is computable on $I_{\mathbb{Q}}$, \mathcal{P} is a Π_1^0 class.

By the “low for z basis theorem” of [8], z is Martin-Löf random, and hence computably random, relative to some member $\langle \eta_0, \eta_1 \rangle$ of \mathcal{P} . Thus, by relativizing (II) to both η_0 and η_1 , we see that f_i is pseudo-differentiable at z for $i = 0, 1$. This implies that f is pseudo-differentiable at z .

By Fact 6.2 this also provides the implication (i) \rightarrow (ii) of Theorem 5.7.

(III). We adapt the proof of Theorem 5.1 to the new setting. For any rational $p > 0$, let

$$\mathcal{C}(p) = \{z : \forall t > 0 \exists a, b [a \leq z \leq b \text{ \& } 0 < b - a \leq t \text{ \& } S_f(a, b) < p],$$

where t, a, b range over rationals. Since f is computable on $I_{\mathbb{Q}}$, the set

$$\{z : \exists a, b [a \leq z \leq b \text{ \& } 0 < b - a \leq t \text{ \& } S_f(a, b) < p\}$$

is a Σ_1^0 set uniformly in t . Then $\mathcal{C}(p)$ is Π_2^0 uniformly in p . Furthermore, $\mathcal{D}f(z) < p \Rightarrow z \in \mathcal{C}(p) \Rightarrow \mathcal{D}f(z) \leq p$. The sets $\tilde{\mathcal{C}}(q)$ are defined analogously, and similar observations hold for them.

Now, in order to show that the set of reals z at which f fails to be pseudo-differentiable is a Σ_3^0 null set, we may conclude the argument as before with the notations $\mathcal{C}(p), \tilde{\mathcal{C}}(q)$ in place of $\underline{\mathcal{C}}(p), \overline{\mathcal{C}}(q)$. \square

6.3. Future directions. We discuss some current research and open problems.

Further algorithmic randomness notions. 2-randomness has not yet been characterized via differentiability of effective functions. One can also attempt to adapt the results on computable randomness in Sections 3 and 4 to the subrecursive case, and in particular to polynomial time randomness. There is an extensive theory of polynomial time computable functions on the unit interval; see for instance near the end of [21].

Values of the derivative. If f is a computable function of bounded variation, then $f'(z)$ exists for all Martin-Löf random reals z by Theorem 5.7. A. Pauly (2011) has asked what can be said about effectivity properties of the derivative as a function on the Martin-Löf random reals. For instance, is it layerwise computable in the sense of Hoyrup and Rojas [12]? Similar questions can be asked about other randomness notions and classes of functions.

Extending the results to higher dimensions. Currently several researchers investigate extensions of the results in this paper to higher dimensions. Already Pathak [17] showed that a weak form of the Lebesgue differentiation theorem similar to Cor. 4.12 holds for Martin-Löf random points in the n -cube $[0, 1]^n$. The above-mentioned work in progress of Rute, and Pathak, Simpson, and Rojas might strengthen this to Schnorr random points in the n -cube. On the other hand, functions of bounded variation can be defined in higher dimension [1, p. 378], and one might try to characterize Martin-Löf randomness in higher dimensions via their differentiability. For weak 2-randomness, new work of Nies and Turetsky yields the analog of Theorem 5.1 in higher dimensions.

Recall from the discussion after Corollary 4.10 that computable randomness can be characterized via differentiability of computable Lipschitz functions. Call a point $x = (x_1, \dots, x_n)$ in the n -cube computably random if no computable martingale succeeds on the binary expansions of x_1, \dots, x_n joined in the canonical way (alternating between the sequences). Differentiability of Lipschitz functions in higher dimensions has been studied in classical analysis. An obvious question is whether computable randomness is also equivalent to differentiability at x of all computable Lipschitz functions in higher dimensions.

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